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Coulomb integrals and conformal blocks in the AdS_3-WZNW model

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Abstract: We study spectral flow preserving four-point correlation functions in the AdS_3-WZNW model using the Coulomb gas method on the sphere. We present a multiple integral realization of the conformal blocks and explicitly compute amplitudes involving operators with quantized values of the sum of their spins, i.e., requiring an integer number of screening charges of the first kind. The result is given as a sum over the independent configurations of screening contours yielding a monodromy invariant expansion in powers of the worldsheet moduli. We then examine the factorization limit and show that the leading terms in the sum can be identified, in the semiclassical limit, with products of spectral flow conserving three-point functions. These terms can be rewritten as the m-basis version of the integral expression obtained by J. Teschner from a postulate for the operator product expansion of normalizable states in the H_3^+-WZNW model. Finally, we determine the equivalence between the factorizations of a particular set of four-point functions into products of two three-point functions either preserving or violating spectral flow number conservation. Based on this analysis we argue that the expression for the amplitude as an integral over the spin of the intermediate operators holds beyond the semiclassical regime, thus corroborating that spectral flow conserving correlators in the AdS_3-WZNW model are related by analytic continuation to correlation functions in the H_3^+-WZNW model.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry, Sigma Models

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1 Introduction

In this article we continue the task, started in [1], of computing correlation functions in the AdS_3-WZNW model within the Coulomb gas approach. In our first paper we used the Wakimoto representation to evaluate both the spectral flow conserving and violating three-point functions of this theory on the sphere and we showed that a proper analytic continuation to non-integer numbers of screening operators gives amplitudes in full agreement with the exact results previously obtained in [2–4]. Here we focus on the spectral flow conserving four-point functions.

Among the many motivations for considering these correlators we can mention their applications to string theory on AdS_3 and the AdS/CFT correspondence as well as further examining the AdS_3-WZNW model as a prototype of non-rational conformal field theory (CFT) with affine Lie algebra symmetry, having close connections with Liouville theory as well as with two and three dimensional gravity. Renewed interest in the study of the conformal blocks originates in the recent developments presented in [5] (see also [6]) where it is conjectured that the conformal blocks of Liouville theory are related to the Nekrasov partition function of a certain class of $\mathcal{N} = 2$ superconformal field theories [7, 8].

Most of what is known about the AdS_3-WZNW model is based on the analytic continuation from its better understood Euclidean counterpart. The solution of the H_3^+-WZNW model on the sphere was achieved in [2, 3] through a generalization of the chiral bootstrap program and more recently it was further explored by means of its relation to Liouville theory [9]. Specifically, it was proved that arbitrary correlation functions on the sphere can be expressed in terms of correlators in Liouville theory which is so far the best understood example of non-rational CFT [10–12]. However, there are many subtleties in the analytic continuation relating the H_3^+ and the AdS_3 models. In particular, the spectral flow automorphism of the latter is a highly non-trivial feature determining a fundamental problem in the application of the bootstrap program. Namely, while the contributions of primary states to the operator product expansion (OPE) in the H_3^+ model are sufficient to complete this program, the descendants not being strictly necessary, the spectral flow operation generates new representations in which the conformal weights are not bounded below, and thus the implementation of the bootstrap approach in the AdS_3 model requires a better understanding of the interplay between different spectral flow sectors and possibly the explicit computation of correlation functions involving affine descentant fields.

Observables in the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) \equiv H_3^+$-WZNW model are given by normalizable functions on the hyperbolic space, having the following form in terms of Poincaré coordinates $(\phi, \gamma, \overline{\gamma})$:

$$\Phi^j(x,\overline{x}|z,\overline{z}) = \frac{1+2j}{\pi}\left(e^{-\sqrt{\frac{1}{2(k-2)}}\phi} + |\gamma - x|^2 e^{\sqrt{\frac{1}{2(k-2)}}\phi}\right)^{2j}, \tag{1.1}$$

where k is the affine level of the algebra, j labels the spin of the state, $(z,\overline{z})$ are the worldsheet coordinates and $(x,\overline{x})$ keep track of the $\mathrm{SL}(2,\mathbb{C})$ quantum numbers. The Hilbert space of this theory is the direct sum of its affine representations with primary states having conformal weights given by $\Delta_j = j(j+1)\rho$, where $j \in \mathcal{P}^+ \equiv -1/2 + i\mathbb{R}_{>0}$ and $\rho = -(k-2)^{-1}$ [13].

Two- and three-point functions involving these operators were obtained in [2, 3] solving the differential equations satisfied by degenerate fields of admissible representations. Based on a proposal for the OPE of normalizable primary fields and on a rigorous treatment in the mini-superspace approximation [14], an expression for the four-point function was also presented in [2, 3]. It involves an integral over a continuous family of solutions of the Knizhnik-Zamolodchikov (KZ) equation, namely,

$$\langle \Phi^{j_1}(0)\Phi^{j_2}(x,\overline{x}|z,\overline{z})\Phi^{j_3}(1)\Phi^{j_4}(\infty)\rangle = \int_{\mathcal{P}} dj\, D(j_1,j_2,j)D(j_4,j_3,j)B(j)^{-1}|\mathcal{F}_j(x|z)|^2, \tag{1.2}$$

where $\mathcal{P} \equiv -1/2 + i\mathbb{R}$, $D(j_1,j_2,j)$ and $D(j_4,j_3,j)$ are the structure constants, $B(j)$ is the propagator of the intermediate state[1] and the conformal blocks have the following expansion:

$$\mathcal{F}_j(x|z) = z^{\Delta_j - \Delta_{j_1} - \Delta_{j_2}} x^{j-j_1-j_2} \sum_{n=0}^{\infty} f_n(x) z^n. \tag{1.3}$$

Substituting this expression into the KZ equation it is found that $f_0(x)$ obeys the hypergeometric equation so that, after imposing monodromy invariance, it is univocally determined.[2]

[1] These functions are explicitly given below in (2.4) and (3.8).

[2] Actually, there are two well-defined independent contributions to the conformal blocks related by the reflection symmetry and monodromy invariance requires including both of them. The extension of the domain of integration from $\mathcal{P}^+$ to $\mathcal{P}$ allows to keep only one of these contributions.

All other $f_n(x)$ can be iteratively computed as stated in [3].

Equation (1.2) holds for operators with spins in the following domain:

$$\begin{cases} |\mathrm{Re}(j_1+j_2+1)| < \frac{1}{2}, & |\mathrm{Re}(j_3+j_4+1)| < \frac{1}{2}, \\ |\mathrm{Re}(j_1-j_2)| < \frac{1}{2}, & |\mathrm{Re}(j_3-j_4)| < \frac{1}{2}. \end{cases}$$

For other values of the spins there are poles in the integrand that hit the contour of integration and the four-point function must be defined by analytic continuation. Crossing symmetry was shown to follow from similar properties of a related five-point function in Liouville theory [15]. The amplitude (1.2) was further studied in the context of string theory on AdS_3 in [4] where, after integrating over the moduli space of the worldsheet, it was written as a sum of products of three-point functions summed over intermediate states lying in the physical spectrum.

The Hilbert space of the AdS_3-WZNW model [16] is very different from that of the Euclidean model. It decomposes into direct products of the normalizable continuous and highest-weight discrete representations of the universal cover of the affine SL$(2,\mathbb{R})$ algebra and their spectral flow images, namely, $\hat{\mathcal{C}}_j^{\alpha,w} \otimes \hat{\mathcal{C}}_j^{\alpha,w}$ with $j \in \mathcal{P}^+$ and $\alpha \in [0,1)$, and $\hat{\mathcal{D}}_j^{-,w} \otimes \hat{\mathcal{D}}_j^{-,w}$ with $-(k-1)/2 < j < -1/2$. The spectral flow parameter w is an integer number. All the states in the spectrum, except those lying in the unflowed continuous representations, correspond to non-normalizable operators in the H_3^+-WZNW model. In order to deal with highest-weight as well as spectral flowed representations it is convenient to work in a basis where the generators $J_0^3, \overline{J}_0^3$ are diagonalized. This is the so-called m-basis. Unflowed operators in the m-basis are related to (1.1) through the following integral transform:

$$\Phi^j_{m,\overline{m}}(z,\overline{z}) = \int d^2x\, x^{j-m}\overline{x}^{j-\overline{m}}\Phi^{-1-j}(x,\overline{x}|z,\overline{z}), \tag{1.4}$$

where m, $\overline{m}$ represent the eigenvalues of J_0^3, $\overline{J}_0^3$, respectively, and $m-\overline{m} \in \mathbb{Z}$. States in $\hat{\mathcal{D}}_j^{-,w}$ ($\hat{\mathcal{C}}_j^{\alpha,w}$) have $m,\overline{m} = j - \mathbb{N}_0$ ($m,\overline{m} = \alpha + \mathbb{Z}$). The spectral flow images of the primary states are obtained from (1.4) acting with the spectral flow operators $\Phi^{-k/2}_{\pm k/2,\pm k/2}$ [4, 16] and they have conformal weight $\hat{\Delta}_{j,m,w} = \Delta_j - mw - kw^2/4$.

Definite expressions for two- and three-point correlation functions of unflowed operators were given in [17] performing the integral transform from the x-basis results of the H_3^+-WZNW model and analytically continuing the kinematical parameters. The accuracy of the analytic continuation is supported by the fact that it leads to the well-known fusion rules for admissible representations [2] and to the classical tensor products of representations of SL$(2,\mathbb{R})$ [17].

Concerning the applications to string theory, this analytic continuation was the starting point for a physical interpretation of the worldsheet correlation functions in terms of correlators in the boundary CFT and for the analysis of the factorization of four-point functions involving unflowed short string states in [4]. Amplitudes involving spectral flowed operators were evaluated transforming to the m-basis the two- and three-point functions of the H_3^+-WZNW model and acting with the spectral flow operator. This process was applied, in particular, to obtain the $w = 1$ three-point function from a special four-point function containing one spectral flow operator.[3] The problem with applying this procedure to (1.2) is that the KZ equation implies the existence of singularities at $z = 0$, 1, x and ∞, which together with monodromy invariance require that the amplitude behaves as

$$\langle \Phi^{j_1}(0)\Phi^{j_2}(x,\overline{x}|z,\overline{z})\Phi^{j_3}(1)\Phi^{j_4}(\infty)\rangle \sim |z-x|^{2(k+j_1+j_2+j_3+j_4)},$$

and, thus, the expansion in (1.3) converges, in principle, only for $|z| < |x|$. Integrals (1.4) transforming to the m-picture can be done either term by term in the expansion in powers of z or with the full correlator obtained by summing all the descendant contributions, but it is not clear that summation and integration will commute or that the sum of integrals over x will converge.

In the sequel we present an independent derivation of the four-point function in the AdS_3-WZNW model directly in the m-basis. This basis has the advantage that correlators of fields with different amount of spectral flow can be treated simultaneously, i.e., all w-conserving amplitudes are the same except for a known factor depending on the insertion points of the vertex operators. We use the Coulomb gas method, which provides a well defined framework within which it should be possible to address this question.

Unlike the successful applications to the minimal models [18] and the SU(2)-WZNW model for operators with half-integer spins [19], the scope of the background charge method in theories with continuous sets of primary fields appears to be limited because they necessarily require non-integer numbers of screening operators. The basic difficulty in going away from half integer SU(2)-like spins is that one no longer deals with degenerate fields satisfying null vector equations. A related problem arose in the evaluation of amplitudes involving operators with rational spins in admissible representations of the SU(2)-WZNW model [19], which could not be accomplished due to the necessity of considering screening currents with rational powers of the ghost fields and the related ambiguity arising in the analytic continuation to non-integer numbers of screening charges.[4] Nevertheless, the formalism has played an important role in the resolution of Liouville theory where an analytic continuation for the three-point function was originally defined in [23] (see also [10, 24, 25]). Similarly, it was shown that the multiple Coulomb integrals define the residues of the on-mass-shell three-point functions not only in Liouville theory but also in Toda CFT [26]. More recently, the suggestion in [5] relating the conformal blocks in Liouville theory and Nekrasov's partition functions revives the longstanding idea that all conformal field theories can be effectively described in the free field formalism [27] because both Dotsenko-Fateev integrals and Nekrasov's functions provide a basis for generalized hypergeometric integrals.

In the case of the AdS_3-WZNW model full agreement was found in [1][5] among the exact three-point functions, both preserving and violating w-number conservation, and those computed via the free field approach for generic values of j. The analytic continuation to non-integer numbers of screening charges was performed in [1] by noticing that the

[3] A related computation was performed within the Coulomb gas formalism in [1].

[4] See also [20–22] for alternative approaches using free field representations.

[5] See [28, 29] for previous related work.

coefficients in the discrete sums arising from the contractions of the β-γ ghost fields can be written in terms of hypergeometric functions which have to be supplemented with an extra contribution determined by monodromy invariance. Defining an unambiguous analytic continuation procedure should open up the possibility of studying arbitrary correlation functions using the Coulomb gas picture for general spins and for any real value of the algebra level. In this paper we move one step forward and extend the techniques developed in our first work to address the computation of w-conserving four-point functions. In particular, we show that there is an alternative representation of the discrete sums arising from the monodromy invariant combination of chiral and anti-chiral conformal blocks in terms of an integral reproducing the m-basis expression which is obtained after applying the transformation (1.4) to (1.2).

As mentioned above, (1.2) was obtained from the OPE of normalizable states in the H_3^+-WZNW model applying the factorization ansatz. But the OPE proposed in [2, 3] would yield an incorrect zero answer if used to compute, for example, w-violating three-point functions in the AdS_3 model; in other words, relaxing the semiclassical approximation in the Lorentzian model is more elusive than in the Euclidean one. Indeed, it was argued in [30] (see also [31]) that a modified OPE should be considered in the former theory including both w-preserving and non-preserving structure constants and it was pointed out that this prescription gives fusion rules of physical states consistent with the spectral flow symmetry and determining the closure of the operator algebra on the Hilbert space of the theory. Consequently, the factorization ansatz would lead to a modified expression for the four-point functions containing both sets of structure constants. However, relying on a plausible but hypothetical identity between two sets of four-point functions, it was shown in [30] that both channels give equivalent contributions for certain w-conserving amplitudes and it was argued that this must also be the case for all w-conserving four-point functions. Here we complete the proof of that identity using the Coulomb gas approach. Actually we show that the Coulomb integral realizations of these two sets of amplitudes agree, thus providing new evidence to support the claim in [30]. This also allows us to conjecture that the results obtained for the four-point functions in the semiclassical limit hold for generic affine level.

The paper is organized as follows. In section 2 we compute spectral flow conserving four-point functions using the Coulomb gas method for spin configurations requiring an integer number of screening operators. In section 3 we examine the factorization limit and show that the leading terms in the expansion of the amplitude in powers of the worldsheet moduli can be written, in the semiclassical regime, as the m-basis version of Teschner's integral expression for the H_3^+-WZNW model. We also prove an identity between two sets of four-point functions which allows to show that the factorization into spectral flow conserving or violating three-point functions give equivalent contributions to these amplitudes and to suggest that the results obtained for the analytic continuation hold for arbitrary affine level. A summary and discussions are included in section 4. Some technical details and other computations are contained in the appendices.

2 Coulomb gas computation of four-point functions

In this section we evaluate w-conserving four-point functions involving spectral flow images of primary fields in the AdS_3-WZNW model using the Coulomb gas method.

Within this formalism, the relevant expectation values are of the form (see [1]):

$$A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \\ w_1, w_2, w_3, w_4 \end{bmatrix} = \left\langle \prod_{i=1}^{4} V_{m_i,\overline{m}_i}^{j_i,w_i}(z_i,\overline{z}_i) \prod_{a=1}^{N_+} \eta^+(\zeta_a^+,\overline{\zeta}_a^+) \prod_{b=1}^{N_-} \eta^-(\zeta_b^-,\overline{\zeta}_b^-) \mathcal{Q}_1^{s_1}\mathcal{Q}_2^{s_2} \right\rangle, \quad (2.1)$$

where the vertices are given by

$$V_{m,\overline{m}}^{j,w}(z,\overline{z}) = \left[e^{(j-m-w)u(z)}e^{-i(j-m)v(z)} \times c.c.\right] e^{\sqrt{\frac{2}{k-2}}\left(j+\frac{k-2}{2}w\right)\phi(z,\bar{z})}, \quad (2.2)$$

the screening operators of the first and second kind are, respectively,

$$\mathcal{Q}_1 = \int d^2y \, \left[\partial v(y) e^{-u(y)+iv(y)} \times c.c.\right] e^{-\sqrt{\frac{2}{k-2}}\phi(y,\overline{y})},$$

and

$$\mathcal{Q}_2 = \int d^2y \, \left[\partial v(y) e^{-u(y)+iv(y)} \times c.c.\right]^{k-2} e^{-\sqrt{2(k-2)}\phi(y,\overline{y})},$$

and the spectral flow vertices act as picture changing operators for the spectral flow sectors and have the following form:

$$\eta^+(\zeta,\overline{\zeta}) = \frac{1}{\pi\Gamma(0)}\left[e^{(k-2)u(\zeta)}e^{-i(k-1)v(\zeta)} \times c.c.\right] e^{\sqrt{2(k-2)}\phi(\zeta,\overline{\zeta})},$$
$$\eta^-(\zeta,\overline{\zeta}) = \frac{1}{\pi\Gamma(0)}\left[e^{iv(\zeta)} \times c.c.\right].$$

These spectral flow operators were introduced in [1] where it was proved that they reproduce, when inserted into multi-point amplitudes, the prescription proposed in [32] and applied in [4] to compute correlators involving spectral flowed states.

Here and thereafter "*c.c.*" indicates that all the variables have to be replaced by the barred ones.

The vertex operators (2.2) are related, in the unflowed case and in the large-ϕ limit, to those defined by (1.4) through

$$\Phi_{m,\overline{m}}^{j}(z,\overline{z}) = V_{m,\overline{m}}^{j}(z,\overline{z}) + B(-1-j)c_{m,\overline{m}}^{-1-j}V_{m,\overline{m}}^{-1-j}(z,\overline{z}), \quad (2.3)$$

where

$$B(j) = -\frac{1+2j}{\pi}\,\nu^{1+2j}\frac{\Gamma(1-\rho(1+2j))}{\Gamma(1+\rho(1+2j))}, \qquad \nu = \pi\frac{\Gamma(1+\rho)}{\Gamma(1-\rho)}, \quad (2.4)$$

and

$$c_{m,\overline{m}}^{j} = \pi\gamma(1+2j)\,\frac{\Gamma(-j-m)\Gamma(-j+\overline{m})}{\Gamma(1+j-m)\Gamma(1+j+\overline{m})}, \qquad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$

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The charge asymmetry conditions for the free field expectation values in (2.1) are given by

$$\sum_{n=1}^{4} j_n = s_1 + (k-2)\left[s_2 - \frac{N_+ + N_-}{2}\right] - 1\,, \tag{2.5}$$

$$\sum_{n=1}^{4} m_n = \sum_{n=1}^{4} \overline{m}_n = \frac{k}{2}\,(N_+ - N_-),$$

$$\sum_{n=0}^{4} w_n = N_- - N_+. \tag{2.6}$$

As expected, for w-conserving amplitudes we must take $N_+ = N_-$.

Notice from (2.5) that $\mathcal{Q}_2$, which only makes sense for $k \in \mathbb{N}_{>2}$, screens exactly the charge carried by a couple of spectral flow operators of each kind. On the other hand, since correlation functions in the m-basis depend on the total w-number and the only change in w-conserving correlators involving states in different spectral flow sectors is contained in the powers of the worldsheet coordinates, assuming $w_i = 0$ for $i = 1, \ldots, 4$, does not imply any loss of generality and we can further take $N_+ = N_- = 0$.

Only correlators with vertices requiring positive integer numbers of screenings, namely, $s_1, s_2 \in \mathbb{N}_0$, can be directly computed in this formalism. Correlation functions involving operators in continuous representations or their spectral flow images cannot be considered at once in this picture because one cannot choose the imaginary parts of the spins in both terms of (2.3) so that they add up to zero in all the terms of the full amplitude. Instead, the second term of the vertex operators creating states in discrete representations vanishes[6] and therefore, the number of charges needed to screen four operators in discrete series turns out to be negative, due to the values of the spins. Negative powers of screenings have been considered in Liouville theory [24, 25] and it was shown that there exists a consistent extension of the formalism to deal with this situation. Alternatively, one can use the reflection symmetry in order to work with positive numbers of screenings. In the sequel we adopt the latter option. In conclusion, only certain states with particular spin values in discrete representations can satisfy equation (2.5) and results for generic configurations require analytic continuation. Therefore, we can take $s_2 = 0$ without loosing generality, and thus $k \in \mathbb{R}_{>2}$.

Summarizing, in this section we evaluate four-point correlation functions involving operators in the unflowed principal discrete representations with $s_2 = 0$, $N_+ = N_- = 0$, namely,

$$A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{bmatrix} \equiv \Gamma(-s)\left\langle V^{j_1, w_1=0}_{m_1, \overline{m}_1}(0) V^{j_2, w_2=0}_{m_2, \overline{m}_2}(z, \overline{z}) V^{j_3, w_3=0}_{m_3, \overline{m}_3}(1) V^{j_4, w_4=0}_{m_4, \overline{m}_4}(+\infty)\mathcal{Q}_1^s\right\rangle, \tag{2.7}$$

[6] See [33] for a discussion on the different asymptotic behaviours of operators in highest-weight or continuous representations.

where $s \equiv s_1 = j_1 + \cdots + j_4 + 1 \in \mathbb{N}_0$. The spectral flow labels in the arguments on the l.h.s. are omitted for short,[7] w refers to the total spectral flow number of the amplitude and global conformal invariance was used in order to set $z_1, \overline{z}_1 = 0$, $z_2 = z$, $\overline{z}_2 = \overline{z}$, $z_3, \overline{z}_3 = 1$ and $z_4, \overline{z}_4 = \infty$. The factor $\Gamma(-s)$ arises from the integration of the zero modes of $\phi(z,\overline{z})$; by abuse of notation, we also denote the vertex operators as $V^{j,w}_{m,\overline{m}}$ after performing this integration.

The corresponding field contractions give:

$$A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{bmatrix} = \Gamma(-s)|z|^{4j_1j_2\rho}|1-z|^{4j_2j_3\rho}\int [dy]\prod_{i=1}^{s}|y_i|^{-4j_1\rho}|z-y_i|^{-4j_2\rho} \\ \times|1-y_i|^{-4j_3\rho}\prod_{j>i}^{s}|y_i-y_j|^{4\rho}\left[\frac{1}{P}\,\partial_1\cdots\partial_s P\times c.c.\right], \tag{2.8}$$

where $[dy]$ is a shorthand for $\prod_{i=1}^{s} d^2y_i$, $\partial_i \equiv \partial/\partial y_i$, and

$$P = \prod_{i=1}^{s} y_i^{-j_1+m_1}(z-y_i)^{-j_2+m_2}(1-y_i)^{-j_3+m_3}\prod_{j>i}^{s}(y_i-y_j) \tag{2.9}$$

is the contribution from the (holomorphic) ghost system. Recall that the field ϕ and the free bosons u and v, which bosonize the usual β-γ ghost system, have propagators:

$$\langle u(z)u(w)\rangle = \langle v(z)v(w)\rangle = \langle\phi(z)\phi(w)\rangle = -\log(z-w),$$

and similar expressions hold for the anti-holomorphic components.

In the next subsection we compute the ghost contributions and then we proceed to the evaluation of the Coulomb integrals.

2.1 Contributions from the ghost system

It is convenient to recall the definition of the Vandermonde determinant:

$$\prod_{i=1}^{s}\prod_{j>i}^{s}(y_i-y_j) = \det\left(y_i^{j-1}\right),$$

and use it to rewrite (2.9) as follows:

$$P = \det\left[(z-y_i)^{-j_2+m_2}(1-y_i)^{-j_3+m_3}y_i^{j-1-j_1+m_1}\right].$$

Taking derivatives as

$$\partial_1\cdots\partial_s P = \det\left\{\partial_i\left[(z-y_i)^{-j_2+m_2}(1-y_i)^{-j_3+m_3}y_i^{j-1-j_1+m_1}\right]\right\} \\ = \left[\prod_{i=1}^{s} y_i^{-j_1+m_1-1}(z-y_i)^{-j_2+m_2-1}(1-y_i)^{-j_3+m_3-1}\right]\det\left[\sum_{l=0}^{2}\ell_l^j(z)y_i^{j-1+l}\right],$$

[7] We shall explicitly write these labels in section 3, when the spectral flow numbers of the operators become relevant.

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we can write

$$\frac{1}{P}\partial_1\cdots\partial_s P=\left[\prod_{i=1}^{s} y_i^{-1}(z-y_i)^{-1}(1-y_i)^{-1}\right]\frac{\det\left[\sum_{l=0}^{2}\ell_l^j(z)y_i^{j-1+l}\right]}{\det\left(y_i^{j-1}\right)}, \tag{2.10}$$

where we have introduced:

$$\begin{cases}\ell_0^j(z)=(j-1-j_1+m_1)z,\\ \ell_1^j(z)=1-j+j_1+j_2-m_1-m_2+(1-j+j_1+j_3-m_1-m_3)z,\\ \ell_2^j(z)=j-1-j_1-j_2-j_3+m_1+m_2+m_3.\end{cases}$$

Notice that the entries of the matrix in the numerator are three term polynomials in the screening variables with powers exceeding in 2, 1 and 0 units the corresponding powers of the entries of the matrix in the denominator. Using the multilinearity of the determinants and performing all the distributions we get

$$\begin{aligned}\det\left[\sum_{l=0}^{2}\ell_l^j(z)y_i^{j-1+l}\right] &= \sum_{\lambda\in[0,2]^s}\det\left[\ell^j_{\lambda_{s+1-j}}(z)y_i^{j-1+\lambda_{s+1-j}}\right]\\ &=\sum_{\lambda\in[0,2]^s}\left[\prod_{j=1}^{s}\ell^j_{\lambda_{s+1-j}}(z)\right]\det\left(y_i^{j-1+\lambda_{s+1-j}}\right),\end{aligned}$$

where the sum index λ is the s-tuple whose components give the excedent in power of the matrix elements with respect to those of the Vandermonde determinant, in the inverse order. Thus the quotient of determinants in (2.10) looks like the one defining Schur polynomials, except for the fact that λ is not a partition but an s-tuple with entries taking the values 0, 1 and 2. Nevertheless, we shall show that it is possible to rewrite (2.10) so as to sum only over partitions.

The s-tuples of the form $(\dots,0,1,\dots)$ or $(\dots,1,2,\dots)$ give no contribution to the quotient in (2.10) since the determinant in the numerator vanishes, and thus the sum is over partitions except for the case of s-tuples of the form $(\dots,0,2,\dots)$. But in this case neither the s-tuples of the form $(\dots,0,2,2,\dots)$ nor those of the form $(\dots,0,0,2,\dots)$ contribute because, again, the determinant in the numerator vanishes. Consequently, only the following s-tuples are relevant:

$$\lambda=(2,\dots,2,1,\dots,1,0,2,1,\dots,1,0,2,1,\dots,1,0,\dots,0)\,. \tag{2.11}$$

Since the interchange of two columns in a determinant only changes its overall sign, the quotient of determinants in (2.10) associated with the s-tuple (2.11) is equal to the Schur polynomial associated to the partition $(2,\dots,2,1,\dots,1,0,\dots,0)$ up to a factor (± 1) depending on the number of times the subsequence "$\dots,0,2,\dots$" is replaced by "$\dots,1,1,\dots$" in (2.11) in order to obtain a partition. This implies that we can actually write

$$\frac{\det\left[\sum_{l=0}^{2}\ell_l^j(z)y_i^{j-1+l}\right]}{\det\left(y_i^{j-1}\right)}=\sum_{\lambda}C_\lambda(z)s_\lambda(y_1,\dots,y_s),$$

where now the sum is over partitions of length s and entries $0, 1$ or 2. These partitions are characterized by two integer numbers, say n and r, denoting the number of times the entries 2 and 1 appear, respectively. Let us write $C_{nr}(z)$ instead of $C_\lambda(z)$ and $s_{nr}(y_1, \ldots, y_s)$ instead of $s_\lambda(y_1, \ldots, y_s)$. We then have:

$$\frac{1}{P}\,\partial_1 \cdots \partial_s P = \left[\prod_{i=1}^{s} y_i^{-1}(z-y_i)^{-1}(1-y_i)^{-1}\right] \sum_{n=0}^{s}\sum_{r=0}^{s-n} C_{nr}(z) s_{nr}(y_1, \ldots, y_s),$$

so that the four-point function (2.8) may be rewritten as

$$A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{bmatrix} = \Gamma(-s)|z|^{4j_1j_2\rho}|1-z|^{4j_2j_3\rho} \sum_{n,\overline{n}=0}^{s}\sum_{r,\overline{r}=0}^{s-n} \left[C_{nr}(z) \times c.c.\right] \mathcal{J}_{nr,\overline{nr}}(z,\overline{z}),$$

where $\mathcal{J}_{nr,\overline{nr}}(z,\overline{z})$ are the following generalized Selberg complex integrals:

$$\begin{aligned}\mathcal{J}_{nr,\overline{nr}}(z,\overline{z}) \;=\; &\int [dy] \prod_{i=1}^{s} |y_i|^{-4j_1\rho-2}|z-y_i|^{-4j_2\rho-2} \times |1-y_i|^{-4j_3\rho-2} \prod_{i<j}^{s} |y_i-y_j|^{4\rho} \\ &\times s_{nr}(y_1, \ldots, y_s) s_{\overline{nr}}(\overline{y}_1, \ldots, \overline{y}_s).\end{aligned} \tag{2.12}$$

Therefore, the problem has been reduced to two independent calculations, namely, obtaining the coefficients $C_{nr}(z)$ and performing the computation of the Coulomb integrals $\mathcal{J}_{nr,\overline{nr}}(z,\overline{z})$. The coefficients $C_{nr}(z)$ are computed in the appendix A.1. They involve complicated hypergeometric-like expansions with polynomials as arguments (see (A.5)). In the following subsection we compute them assuming that a highest-weight state is inserted at $z_1, \overline{z}_1 = 0$; many simplifications occur in this case and this allows us to deal with the Coulomb integrals in subsection 2.3.

2.2 Evaluation of m-dependent coefficients: one highest-weight state

Explicitly evaluating the terms contributing to $C_{nr}(z)$ in (2.10) for different values of n and r, it can be shown that

$$C_{nr}(z) = (-1)^{s-n-r} z^{s-n-r} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-s+n+r+1)} \frac{\Gamma(s-\alpha-\beta-\gamma)}{\Gamma(s-n-\alpha-\beta-\gamma)}\, d_{s-n}(z), \tag{2.13}$$

where we have defined

$$\begin{cases} \alpha = j_1 - m_1, \\ \beta = j_2 - m_2, \\ \gamma = j_3 - m_3, \end{cases}$$

$d_{s-n}(z)$ is the determinant of the matrix $(a_{ij}(z))_{i,j=1}^{s-n}$ with entries given by

$$a_{ij}(z) = \ell_{i-j+1}^{s-n-r+j}(z), \tag{2.14}$$

and we are setting $a_{ij}(z) = 0$ if $|i-j| > 1$.

As we have mentioned, many simplifications occur if the operator inserted at $z_1, \overline{z}_1 = 0$ creates a highest-weight state, so from here on we assume $j_1 = m_1$. In appendix A.1 we present the computation in full generality.

The coefficients $C_{nr}(z)$ vanish when $\alpha = 0$ unless $r = s - n$ and in this case we have

$$C_n(z) \equiv C_{nr=s-n}(z) = \frac{\Gamma(s-\beta-\gamma)}{\Gamma(s-n-\beta-\gamma)}\, d^0_{s-n}(z),$$

where $d^0_{s-n}(z)$ is the determinant of the matrix $(a^0_{ij}(z))^{s-n}_{i,j=1}$ with elements $a^0_{ij}(z) = \ell^j_{i-j+1}(z)$.

Let us denote by $d^0_p(z)$, $p = 1, 2, \ldots, s-n$, the determinant of the matrix $(a^0_{ij}(z))^p_{i,j=1}$ formed by the first p rows and p columns of $(a^0_{ij}(z))^{s-n}_{i,j=1}$. Notice that $d^0_p(z)$ is a polynomial in z of degree p satisfying the following recurrence formula:

$$d^0_p(z) = \ell^p_1(z) d^0_{p-1}(z) - \ell^{p-1}_2(z)\ell^p_0(z) d^0_{p-2}(z),$$

or, more explicitly,

$$d^0_p(z) = [(1-p+\beta) + (1-j+\gamma)z] d^0_{p-1}(z) - (p-2-\beta-\gamma)(p-1) z d^0_{p-2}(z), \qquad (2.15)$$

which follows from the fact that $(a^0_{ij}(z))^p_{i,j=1}$ is a tridiagonal matrix.

The boundary conditions for this recurrence are: $d^0_1(z) = \ell^1_1(z)$ and $d^0_2(z) = \ell^1_1(z)\ell^2_1(z) - \ell^1_2(z)\ell^2_0(z)$.

It can be inductively proved that the solution of (2.15) is given by

$$d^0_p(z) = \frac{\Gamma(\beta+1)}{\Gamma(-p)\Gamma(-\gamma)} \sum_{t=0}^{p} \frac{\Gamma(-p+t)\Gamma(-\gamma+t)}{\Gamma(\beta-p+1+t)} \frac{z^t}{t!}. \qquad (2.16)$$

Finally, noticing that the sum over t can be freely taken to ∞, we can write

$$d^0_{s-n}(z) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-s+n+1)}\, {}_2F_1\left[\begin{array}{c} -s+n, -\gamma \\ \beta-s+n+1 \end{array}\middle|\, z\right],$$

and then,

$$C_n = B_n\, {}_2F_1\left[\begin{array}{c} -s+n, -j_3+m_3 \\ j_2-m_2-s+n+1 \end{array}\middle|\, z\right], \qquad (2.17)$$

where

$$B_n \equiv \frac{\Gamma(s-j_2-j_3+m_2+m_3)\Gamma(j_2-m_2+1)}{\Gamma(s-n-j_2-j_3+m_2+m_3)\Gamma(j_2-m_2-s+n+1)}. \qquad (2.18)$$

On the other hand, since only partitions of the form $(2, \ldots, 2, 1, \ldots, 1)$ appear we can use

$$s_{n,s-n}(y_1, \ldots, y_s) = \left[\prod_{i=1}^{s} y_i\right] \times \alpha^s_n(y_1, \ldots, y_s)\,,$$

where

$$\alpha^s_n(y_1, \ldots, y_s) = \frac{1}{n!(s-n)!} \sum_{\sigma_n} \prod_{i=1}^{n} y_{\sigma_{n(i)}}$$

is an elementary symmetric polynomial and $\alpha_0^s = 1$, to finally conclude that the four-point function involving one highest-weight state is given by

$$A_4^{w=0}\left[\begin{matrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, m_3, m_4 \end{matrix}\right] \tag{2.19}$$
$$= \Gamma(-s)|z|^{4j_1j_2\rho}|1-z|^{4j_2j_3\rho}\sum_{n,\overline{n}=0}^{s}\left\{B_n\,{}_2F_1\left[\begin{matrix} -s+n, -j_3+m_3 \\ j_2-m_2-s+n+1 \end{matrix}\middle| z\right]\times c.c.\right\}\times\mathcal{J}_{n,\overline{n}}(z,\overline{z}),$$

where we have defined

$$\mathcal{J}_{n,\overline{n}}(z,\overline{z}) = \int [dy]\prod_{i=1}^{s}|y_i|^{-4j_1\rho}|z-y_i|^{-4j_2\rho-2}|1-y_i|^{-4j_3\rho-2}\prod_{i<j}|y_i-y_j|^{4\rho}$$
$$\times\ \alpha_n^s(y_1,\ldots,y_s)\,\overline{\alpha}_{\overline{n}}^s(\overline{y}_1,\ldots,\overline{y}_s). \tag{2.20}$$

Notice that, other than in the explicit overall factor, the z-dependence of (2.20) appears in the hypergeometric function and both in the integrand and in the integration domain of $\mathcal{J}_{n,\overline{n}}(z,\overline{z})$. In the next section we analyze this dependence in detail.

2.3 Monodromy invariance and normalization

According to the analysis in [18], the integral $\mathcal{J}_{n,\overline{n}}(z,\overline{z})$ is given by the monodromy invariant combination of chiral and antichiral conformal blocks as

$$\mathcal{J}_{n,\overline{n}}(z,\overline{z}) = \sum_{l=0}^{s} X_l^{n\overline{n}} I_n^l(z) I_{\overline{n}}^l(\overline{z})\,, \tag{2.21}$$

where

$$I_n^l(z) = \int_{\Delta_{s-l}^{(1,\infty)}}\prod_{i=1}^{s-l}dy_i\int_{\Delta_l^{(0,z)}}\prod_{i=1}^{l}dy_{s-l+i}\prod_{i=1}^{s}y_i^{-2j_1\rho}\prod_{i<j}^{s}(y_i-y_j)^{2\rho}\prod_{i=1}^{s-l}(y_i-z)^{-2j_2\rho-1}(y_i-1)^{-2j_3\rho-1}$$
$$\times\prod_{i=1}^{l}(z-y_{s-l+i})^{-2j_2\rho-1}(1-y_{s-l+i})^{-2j_3\rho-1}\alpha_n^s(y_1,\ldots,y_s),$$

the integration domains being the simplex $\Delta_{s-l}^{(1,\infty)} \equiv \{(y_1,\ldots,y_{s-l}) : 1 < y_{s-l} < \cdots < y_1 < +\infty\}$ and $\Delta_l^{(0,z)} \equiv \{(y_{s-l+1},\ldots,y_s) : 0 < y_s < \cdots < y_{s-l+1} < z\}$.

The form (2.21) is diagonal in $I_n^l(z)$ since these functions have diagonal s-channel monodromy (as we show below). The coefficients $X_l^{n\overline{n}}$ are determined from the requirement that the physical four-point function must be monodromy invariant with respect to the analytic continuation over z and $\overline{z}$ around $z,\overline{z}=0$ and around $z,\overline{z}=1$.

Alternatively to $I_n^l(z)$, one may consider the following "unordered" integrals:

$$J_n^l(z) = \int_{(1,\infty)^{s-l}}\prod_{i=1}^{s-l}dy_i\int_{(0,z)^l}\prod_{i=1}^{l}dy_{s-l+i}\prod_{i=1}^{s}y_i^{-2j_1\rho}\prod_{i=1}^{s-l}\,(y_i-z)^{-2j_2\rho-1}(y_i-1)^{-2j_3\rho-1}$$
$$\times\prod_{i=1}^{l}(z-y_{s-l+i})^{-2j_2\rho-1}(1-y_{s-l+i})^{-2j_3\rho-1}\prod_{i<j}^{s}(y_i-y_j)^{2\rho}\alpha_n^s(y_1,\ldots,y_s), \tag{2.22}$$

which are related to $I_n^l(z)$ as

$$J_n^l(z) = \lambda_l(\rho) I_n^l(z),$$

with

$$\lambda_l(\rho) = \prod_{i=1}^{s-l} \frac{\sin(i\pi\rho)}{\sin(\pi\rho)} \prod_{i=1}^{l} \frac{\sin(i\pi\rho)}{\sin(\pi\rho)}\,.$$

The symmetry of $\alpha_n^s(y_1,\ldots,y_s)$ under any permutation of its arguments renders the proof of this statement exactly as in [18]. For given values of n, the functions (2.22) with different values of l are mutually independent. They provide the integral representation for the system of diagonal conformal blocks with respect to the s-channel.

In order to prove that the monodromy group around $z=0$ acts diagonally on the basis $\{J_n^l(z)\}$, we perform the change of variables $y_{s-l+q} = zu_q$ for $q=1,2,\ldots,l$ and get

$$\begin{aligned} J_n^l(z) \;=\; & z^{-2lj_1\rho-2lj_2\rho+\rho l(l-1)} \int_{(1,\infty)^{s-l}} \prod_{i=1}^{s-l} dy_i \int_{(0,1)^l} \prod_{i=1}^{l} du_i \prod_{i=1}^{s-l} y_i^{-2j_1\rho}(y_i-z)^{-2j_2\rho-1} \\ & \times (y_i-1)^{-2j_3\rho-1} \prod_{i<j}^{s-l} (y_i-y_j)^{2\rho} \prod_{q=1}^{l} u_q^{-2j_1\rho}(1-u_q)^{-2j_2\rho-1}(1-zu_q)^{-2j_3\rho-1} \\ & \times \prod_{q<p}^{l} (u_q-u_p)^{2\rho} \prod_{i=1}^{s-l}\prod_{q=1}^{l} (y_i-zu_q)^{2\rho} \alpha_n^s(y_1,\ldots,y_{s-l},zu_1,\ldots,zu_l)\,. \end{aligned} \tag{2.23}$$

It is easy to see that $\alpha_n^s(y_1,\ldots,y_{s-l},zu_1,\ldots,zu_l)$ does not change the overall z-dependence of (2.23) if $n=0,1,2,\ldots,s-l$, but an extra factor z^{l+n-s} appears if $n \geq s-l+1$. However a rotation around $z=0$ gives no additional phase factor since $l+n-s$ is an integer number. After extracting the z-dependence in (2.23), the integral is an analytic function, regular at $z=0$. Consequently, a monodromy transformation around $z=0$ gives

$$J_n^l(z) \to \exp[-2l\pi i(2j_1+2j_2-l+1)\rho] \times J_n^l(z)\,.$$

The following integrals provide an alternative basis for (2.20) [18]:

$$\begin{aligned} \widetilde{J}_n^l(z) = & \int_{(-\infty,0)^{s-l}} \prod_{i=1}^{s-l} dy_i \int_{(z,1)^l} \prod_{i=1}^{l} dy_{s-l+i} \prod_{i=1}^{s-l} (-y_i)^{-2j_1\rho}(z-y_i)^{-2j_2\rho-1}(1-y_i)^{-2j_3\rho-1} \\ & \times \prod_{i=1}^{l} y_{s-l+i}^{-2j_1\rho}(y_{s-l+i}-z)^{-2j_2\rho-1}(1-y_{s-l+i})^{-2j_3\rho-1} \prod_{i>j}^{s} (y_i-y_j)^{2\rho} \alpha_n^s(y_1,\ldots,y_s). \end{aligned}$$

In this case we may prove that this set is a canonical basis for the point $z=1$. To this aim, it is convenient to perform two changes of variables, first $y_i \longrightarrow 1-y_i$ and then

$y_{s-l+q} \longrightarrow (1-z)u_q$, for $q=1,2,\ldots,l$. This gives:

$$\begin{aligned}\widetilde{J}^l_n(z) \;=\; &(1-z)^{-2lj_3\rho-2lj_2\rho-l+l(l-1)\rho}\sum_{n'=0}^{n}(-1)^{n'}\begin{pmatrix} s-n' \\ n-n'\end{pmatrix}\int_{(1,\infty)^{s-l}}\prod_{i=1}^{s-l}dy_i\int_{(0,1)^l}\prod_{i=1}^{l}du_i \\ &\times\prod_{i=1}^{s-l}(y_i-1)^{-2j_1\rho}(y_i-(1-z))^{-2j_2\rho-1}y_i^{-2j_3\rho-1}\prod_{i<j}^{s-l}(y_i-y_j)^{2\rho} \\ &\times\prod_{q=1}^{l}(1-(1-z)u_q)^{-2j_1\rho}u_q^{-2j_3\rho-1}(1-u_q)^{-2j_2\rho-1}\prod_{q<p}^{l}(u_q-u_p)^{2\rho} \\ &\prod_{i=1}^{s-l}\prod_{q=1}^{l}(y_i-(1-z)u_q)^{2\rho}\alpha^s_{n'}(y_1,\ldots,y_{s-l},(1-z)u_1,\ldots,(1-z)u_l),\end{aligned}$$

where we have used the following identity

$$\alpha^s_n(1-w_1,\ldots,1-w_s) = \sum_{n'=0}^{n}(-1)^{n'}\begin{pmatrix} s-n' \\ n-n'\end{pmatrix}\alpha^s_{n'}(w_1,\ldots,w_s), \tag{2.24}$$

which can be proved inductively.

In this case, if $n=0,1,\ldots,s-l$, i.e., $l+n\leq s$, then $l+n'\leq s$ and there is no additional overall $(1-z)$-dependence coming from the elementary symmetric polynomials, as we have seen. If $l+n>s$, for $l+n'>s$ an extra factor $(1-z)^{l+n'-s}$ should be considered. But, again, a loop around $z=1$ gives no non-trivial phase factor since $l+n'-s\in\mathbb{N}_0$. It follows that a monodromy transformation around $z=1$ is given by

$$\widetilde{J}^l_n(z)\to\exp[-2l\pi i(2j_2+2j_3-l+1)\rho]\times\widetilde{J}^l_n(z),$$

and $\{\widetilde{J}^l_n(z)\}$ is, thus, a canonical basis for $z=1$.

Having checked that $\{J^l_n(z)\}$ and $\{\tilde{J}^l_n(z)\}$ are canonical basis for the points $z=0$ and $z=1$, respectively, the computation of the factors $X^{n\overline{n}}_l$ follows as in [18]. Since they do not depend on $n,\overline{n}$ we may write[8]

$$\begin{aligned}X_l\equiv X^{n\overline{n}}_l = \frac{1}{s!}\prod_{i=1}^{s-l}\frac{\sin(i\pi\rho)}{\sin(\pi\rho)}\prod_{i=1}^{l}\frac{\sin(i\pi\rho)}{\sin(\pi\rho)}\prod_{i=0}^{l-1}\frac{\sin(\pi(1-2j_1\rho+i\rho))\sin(\pi(-2j_2\rho+i\rho))}{\sin(\pi(1-2j_1\rho-2j_2\rho+(l-1+i)\rho))} \\ \times\prod_{i=0}^{s-l-1}\frac{\sin(\pi(-2j_3\rho+i\rho))\sin(\pi(1+2j_1\rho+2j_3\rho+2j_2\rho-2\rho(s-1)+i\rho))}{\sin(\pi(1+2j_1\rho+2j_2\rho-2\rho(s-1)+(s-l-1+i)\rho))}.\end{aligned} \tag{2.25}$$

Following a related computation in [19], let us write

$$z^{2j_1j_2\rho}(1-z)^{2j_2j_3\rho}{}_2F_1\left[\begin{matrix}-s+n,-j_3+m_3 \\ j_2-m_2-s+n+1\end{matrix}\,\middle|\,z\right]J^l_n(z)=\lambda_l(\rho)N^l_nf^l_n(z)z^{\gamma^l_n}, \tag{2.26}$$

[8]Recall that these coefficients are defined up to an overall l-independent factor to be determined from the two-point function [18]. Here they are already normalized.

where $f_n^l(z)$ are regular functions of z with $f_n^l(0) = 1$, so that the four-point function can be rewritten as

$$A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, m_3, m_4 \end{bmatrix} = \Gamma(-s)\sum_{l=0}^{s}\sum_{n,\bar{n}=0}^{s} X_l \left\{ z^{\gamma_n^l} B_n N_n^l f_n^l(z) \times c.c. \right\}. \tag{2.27}$$

The expressions for N_n^l and γ_n^l differ, depending on whether $(a) : l+n \leq s$, or $(b) : l+n > s$.

Case (*a*). In this case, we have

$$\gamma_n^l = 2j_1 j_2 \rho - 2l j_1 \rho - 2l j_2 \rho + \rho l(l-1)\,, \tag{2.28}$$

and since it does not depend on n, we will denote it simply by γ_l.

The normalization constant N_n^l is obtained omitting the overall z-dependence in (2.23) and afterwards taking the limit $z \to 0$, namely,

$$\begin{aligned} N_n^l = &\frac{1}{l!(s-l)!}\int_{(0,1)^l}\prod_{i=1}^{l} du_i \prod_{q=1}^{l} u_q^{-2j_1\rho}(1-u_q)^{-2j_2\rho-1}\prod_{q<p}^{l}|u_q-u_p|^{2\rho} \\ &\times \int_{(1,\infty)^{s-l}}\prod_{i=1}^{s-l} dy_i \prod_{i=1}^{s-l} y_i^{-2j_1\rho-2j_2\rho-1+2l\rho}(y_i-1)^{-2j_3\rho-1}\prod_{i<j}^{s-l}|y_i-y_j|^{2\rho}\alpha_n^{s-l}(y_1,\dots,y_{s-l}), \end{aligned} \tag{2.29}$$

where we have used the identity

$$\alpha_n^s(y_1,\dots,y_{s-l},0,\dots,0) = \alpha_n^{s-l}(y_1,\dots,y_{s-l}). \tag{2.30}$$

Let us denote the Selberg integrals in the first line[9] as $S_l(-2j_1\rho+1, -2j_2\rho, \rho)$. The remaining integral can be computed using Aomoto's formula. Indeed, changing variables $y_i \to 1/y_i$ and using the conservation laws (2.5)–(2.6), we can rewrite the last line in (2.29) as

$$\begin{aligned} A_{s-l}^{s-l-n}(-2j_4\rho, -2j_3\rho, \rho) = &\int_{(0,1)^{s-l}}\prod_{i=1}^{s-l} dy_i \prod_{i=1}^{s-l} y_i^{-2j_4\rho-1}(1-y_i)^{-2j_3\rho-1} \\ &\times \prod_{i<j}^{s-l}|y_i-y_j|^{2\rho}\alpha_{s-l-n}^{s-l}(y_1,\dots,y_{s-l}), \end{aligned}$$

where we have used the identity

$$\alpha_n^{s-l}\left(\frac{1}{y_1},\dots,\frac{1}{y_{s-l}}\right) = \left[\prod_{i=1}^{s-l} y_i^{-1}\right]\alpha_{s-l-n}^{s-l}(y_1,\dots,y_{s-l}).$$

Therefore, the normalization constant may be written as

$$N_n^l = \frac{1}{l!(s-l)!}S_l(-2j_1\rho+1, -2j_2\rho, \rho)A_{s-l}^{s-l-n}(-2j_4\rho, -2j_3\rho, \rho). \tag{2.31}$$

[9] Recall that we are using the notation introduced in [1].

Case (b)**.** When $l+n>s$ we have

$$\gamma_n^l = \gamma_l + l + n - s, \tag{2.32}$$

and

$$N_n^l = \frac{1}{l!(s-l)!}\int_{(0,1)^l}\prod_{i=1}^{l}du_i\prod_{q=1}^{l}u_q^{-2j_1\rho}(1-u_q)^{-2j_2\rho-1}\prod_{q<p}^{l}|u_q-u_p|^{2\rho}\alpha_{n+l-s}^{l}(u_1,\dots,u_l)$$
$$\times\int_{(0,1)^{s-l}}\prod_{i=1}^{s-l}dy_i\prod_{i=1}^{s-l}y_i^{-2j_4\rho-1}(1-y_i)^{-2j_3\rho-1}\prod_{i<j}^{s-l}|y_i-y_j|^{2\rho},$$

where we have used

$$\lim_{z\to 0} z^{s-l-n}\alpha_n^s(y_1,\dots,y_{s-l},zu_1,\dots,zu_l) = \left[\prod_{i=1}^{s-l}y_i\right]\alpha_{n+l-s}^{l}(u_1,\dots,u_l), \tag{2.33}$$

and therefore we get

$$N_n^l = \frac{1}{l!(s-l)!}A_l^{l+n-s}(-2j_1\rho+1,-2j_2\rho,\rho)S_{s-l}(-2j_4\rho,-2j_3\rho,\rho). \tag{2.34}$$

Notice that the values of γ_n^l given by (2.32) are always greater than those in (2.28) and thus they do not contribute to the lowest order in the factorization limit.

Using the identities (see [18] and [1]):

$$\mathcal{S}_l(a,b,\rho) = \frac{1}{l!}\prod_{i=1}^{l}\frac{\sin(i\pi\rho)}{\sin(\pi\rho)}\prod_{i=0}^{l-1}\frac{\sin(\pi(a-1+i\rho))\sin(\pi(b-1+i\rho))}{\sin(\pi(a+b-2+(l-1+i)\rho))}S_l(a,b,\rho)^2, \tag{2.35}$$
$$\mathcal{A}_l^{n,\overline{n}}(a,b,\rho) = \frac{1}{l!}\prod_{i=1}^{l}\frac{\sin(i\pi\rho)}{\sin(\pi\rho)}\prod_{i=0}^{l-1}\frac{\sin(\pi(a-1+i\rho))\sin(\pi(b-1+i\rho))}{\sin(\pi(a+b-2+(l-1+i)\rho))}A_l^n(a,b,\rho)A_l^{\overline{n}}(a,b,\rho),$$

and replacing (2.18), (2.25), (2.28), (2.31), (2.32) and (2.34) into (2.27), it follows that the four-point amplitude may be rewritten in the following useful form:

$$A_4^{w=0}\begin{bmatrix} j_1,j_2,j_3,j_4 \\ j_1,m_2,m_3,m_4\end{bmatrix}$$
$$= \Gamma(-s)\sum_{l=0}^{s}|z|^{2\gamma_l}\binom{s}{l}\Bigg[\sum_{n,\overline{n}=0}^{s-l}|B_{s-l-n}|^2\mathcal{S}_l(-2j_1\rho+1,-2j_2\rho,\rho) \tag{2.36}$$
$$\times\mathcal{A}_{s-l}^{n,\overline{n}}(-2j_4\rho,-2j_3\rho,\rho)|f_{s-l-n}^{l}(z)|^2$$
$$+\sum_{n,\overline{n}=1}^{l}z^n\bar{z}^{\bar{n}}|B_{s-l+n}|^2\mathcal{S}_{s-l}(-2j_4\rho,-2j_3\rho,\rho)\mathcal{A}_l^{n,\overline{n}}(-2j_1\rho+1,-2j_2\rho,\rho)|f_{s-l+n}^{l}(z)|^2\Bigg].$$

Recall that this expression holds for integer numbers of screening charges and it involves one highest-weight operator. It is possible to relax the highest-weight restriction using the Campbell-Backer-Hausdorff identity and proceeding as was done for the easier case of the three-point functions in [1, 28]. We shall not perform this tedious calculation here but we show later that the leading terms in the $z, \overline{z} \to 0$ limit of (2.37) can be identified with products of three-point functions from where it is straightforward to see that relaxing the highest-weight condition turns the Selberg integrals into combinations of Aomoto integrals.

In order to have a closed form for the conformal blocks[10] $f_n^l(z)$ one must solve the multiple integrals in (2.22) and this is a difficult task.[11] Even in the simpler cases of Liouville theory or the H_3^+-WZNW model no explicit formula for the conformal blocks is known. However, albeit the existence of a closed expression is unlikely, we shall be able to examine the leading terms in the factorization limit and perform the analytic continuation of (2.37) to generic values of the spins.

3 The factorization limit

In this section we study the leading terms in the factorization limit of the four-point function, *i.e.* we retain only the leading terms in the $z, \overline{z} \to 0$ limit of (2.37) and examine the following expression:

$$\begin{aligned} &\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, m_3, m_4 \end{bmatrix} \\ &\quad \equiv \Gamma(-s) \sum_{l=0}^{s} \sum_{n,\overline{n}=0}^{s-l} |z|^{2\gamma_l} \binom{s}{l} |B_{s-l-n}|^2 \mathcal{S}_l(-2j_1\rho+1, -2j_2\rho, \rho) \mathcal{A}_{s-l}^{n,\overline{n}}(-2j_4\rho, -2j_3\rho, \rho). \end{aligned} \tag{3.1}$$

3.1 Identification of the intermediate channels

The leading powers of z in the factorization of the amplitude of four unflowed states are expected to be of the form $\hat{\Delta}_{j,m,w} - \Delta_{j_1} - \Delta_{j_2}$. In general there are various choices of quantum numbers for which this combination equals γ_l, and then the intermediate channels cannot be unambiguously determined from this equality. No ambiguities arise in the semiclassical regime where only unflowed operators are expected to appear, so that equating $\gamma_l = \Delta_j - \Delta_{j_1} - \Delta_{j_2}$ one can read the possible values of the spin of the intermediate states. They are given by $j \equiv j_0 = -1 - j_1 - j_2 + l$ and $j = -1 - j_0$, in agreement with [19].

Consistently with this identification, let us now show that the sums over n and $\overline{n}$ in (3.2) can be rewritten as products of two $w = 0$ three-point functions divided by the two-point function of the unflowed intermediate state.

[10] Although they are closely related, the functions $f_n^l(z)$ should be distinguished from $f_n(x)$ introduced in (1.2).

[11] Actually, (2.22) is obtained when the amplitude involves one highest-weight state. The most general expression for the conformal blocks when four generic states are considered can be reconstructed replacing $\alpha_n^s(y_1, \dots, y_s)$ by $s_{nr}(y_1, \dots, y_s)$ and the coefficients C_{nr} given in (2.17) by (A.5). In the limit $z, \overline{z} \to 0$, the integrals will reduce to those computed in [34], but the manipulations performed in this section with the elementary symmetric polynomials cannot be easily generalized when arbitrary Schur polynomials are involved.

Recall the expression for the three-point functions given by eq. (3.42) in [1]. Changing labels, this equation may be rewritten as

$$A_3^{w=0}\begin{bmatrix} j_4, j_3, -1-j \\ m_4, m_3, m \end{bmatrix} = \Gamma(-s+l)\left[\frac{\Gamma(1+j_4-m_4)}{\Gamma(-s+l+1+j_4+j_3-m_4-m_3)}\times c.c.\right] \tag{3.2}$$
$$\times\sum_{n,\overline{n}=0}^{s-l}(-1)^{n+\overline{n}}\left[\frac{\Gamma(-s+l+1+j_4+j_3-m_4-m_3+n)}{\Gamma(1-s+l+j_4-m_4+n)}\times c.c.\right]\mathcal{A}_{s-l}^{n,\overline{n}}(-2j_4\rho,-2j_3\rho,\rho),$$

where $j_3+j_4-j=s-l$, $m=-m_3-m_4$ and $\mathcal{A}_l^{n,\overline{n}}(-2j_1\rho,-2j_2\rho,\rho)$ was defined in (2.36). The insertion points of the operators in the three-point functions are taken at $(0,1,+\infty)$ and, as before, we omit the obvious $\overline{m}$-dependence from the arguments for short.

Using the conservation laws for the original four-point function, *i.e.* $j_2=s-1-j_1-j_3-j_4$, $m_2=-j_1-m_3-m_4$, and recalling that

$$B_{s-l-n}=\frac{\Gamma(1+j_4-m_4)}{\Gamma(l+n-s+1+j_4-m_4)}\frac{\Gamma(s-j_3-j_4+m_3+m_4)}{\Gamma(s-j_3-j_4+m_3+m_4-l-n)},$$

the three-point function (3.3) may be rewritten as

$$A_3^{w=0}\begin{bmatrix} j_4, j_3, -1-j \\ m_4, m_3, m \end{bmatrix} = \Gamma(-s+l)\frac{\Gamma(1-s+j_3+j_4-m_3-m_4)}{\Gamma(-s+l+1+j_4+j_3-m_4-m_3)}$$
$$\times\frac{\Gamma(1-s+j_3+j_4-\overline{m}_3-\overline{m}_4)}{\Gamma(-s+l+1+j_4+j_3-\overline{m}_4-\overline{m}_3)}$$
$$\sum_{n,\overline{n}=0}^{s-l}|B_{s-l-n}|^2\mathcal{A}_{s-l}^{n,\overline{n}}(-2j_4\rho,-2j_3\rho,\rho).$$

Therefore, it follows from (3.2) that

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, m_3, m_4 \end{bmatrix} = \Gamma(-s)\sum_{l=0}^{s}|z|^{2\gamma_l}\binom{s}{l}\mathcal{S}_l(-2j_1\rho+1,-2j_2\rho,\rho) \tag{3.3}$$
$$\times\frac{\Gamma(-j_2+m_2+l)\Gamma(-j_2+\overline{m}_2+l)}{\Gamma(-s+l)\Gamma(-j_2+m_2)\Gamma(-j_2+\overline{m}_2)}A_3^{w=0}\begin{bmatrix} j_4, j_3, -1-j \\ m_4, m_3, m \end{bmatrix}.$$

Using the following identity proved in [28]:

$$A_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ j_1, m_2, -m \end{bmatrix} = \Gamma(-l)\frac{\Gamma(-j_2+m_2+l)\Gamma(-j_2+\overline{m}_2+l)}{\Gamma(-j_2+m_2)\Gamma(-j_2+\overline{m}_2)}\mathcal{S}_l(-2j_1\rho+1,-2j_2\rho,\rho),$$

where $j_1+j_2+j+1=l$, (3.3) may be recast as

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, m_3, m_4 \end{bmatrix} = \frac{1}{\Gamma(0)}\sum_{l=0}^{s}|z|^{2\gamma_l}A_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ j_1, m_2, -m \end{bmatrix}A_3^{w=0}\begin{bmatrix} j_4, j_3, -1-j \\ m_4, m_3, m \end{bmatrix}$$
$$=\sum_{l=0}^{s}|z|^{2\gamma_l}A_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ j_1, m_2, -m \end{bmatrix}A_3^{w=0}\begin{bmatrix} j_4, j_3, -1-j \\ m_4, m_3, m \end{bmatrix}A_2^{w=0}\begin{bmatrix} j, -1-j \\ -m, m \end{bmatrix}^{-1},$$

where we have used $\Gamma(-l)l! = (-1)^l\Gamma(0)$ in the first line and the factor $\Gamma(0)$ has been interpreted as the $\delta^2(0)$ from a two-point function in the second line.

At this point we can straightforwardly relax the highest-weight condition of the state at $z_1, \overline{z}_1 = 0$ using the Backer-Campbell-Hausdorff formula as in [28] to finally get:

$$\begin{aligned}&\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{bmatrix} \\ &\quad = \sum_{l=0}^{s} |z|^{2\gamma_l} A_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ m_1, m_2, -m \end{bmatrix} A_3^{w=0}\begin{bmatrix} j_3, j_4, -1-j \\ m_3, m_4, m \end{bmatrix} A_2^{w=0}\begin{bmatrix} j, -1-j \\ -m, m \end{bmatrix}^{-1},\end{aligned} \tag{3.4}$$

where $j = j_0 \equiv -1-j_1-j_2+l$ (alternatively, $j = -1-j_0$) and $m = m_1+m_2 = -m_3-m_4$.

Changing the index $l \to (s-l)$ in (3.4) we get another parametrization of the four-point function that will be important when discussing its analytic continuation below, namely

$$\begin{aligned}&\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{bmatrix} \\ &\quad = \sum_{l=0}^{s} |z|^{2\gamma'_l} A_3^{w=0}\begin{bmatrix} j_1, j_2, -1-j' \\ m_1, m_2, -m \end{bmatrix} A_3^{w=0}\begin{bmatrix} j_4, j_3, j' \\ m_4, m_3, m \end{bmatrix} A_2^{w=0}\begin{bmatrix} -1-j', j' \\ -m, m \end{bmatrix}^{-1},\end{aligned} \tag{3.5}$$

where γ'_l equals γ_l with the replacement $j \to j' = -1-j_3-j_4+l$.

Eq. (3.4) expresses the content of the factorization limit of the four-point functions obtained in the Coulomb gas approach in the semiclassical limit. However, this expression was deduced without making any assumption on the values of k, except for the identification of the intermediate channels with unflowed operators. It is surprising that all the terms in (3.2) can be identified as contributions of $w = 0$ intermediate states because it was shown in [17] that the OPE of unflowed states (when defined as in [3]) contains contributions from operators outside the physical spectrum of the AdS_3-WZNW model and it was argued in [30] that the spectral flow symmetry of the model requires to additionally consider w-violating structure constants. In section 3.3 we elaborate on these issues.

3.2 Analytic continuation

In order to perform the analytic continuation of $\mathbb{A}_4^{w=0}$ for generic external states, notice that the integer nature of the number of screening charges is encoded both in the upper limit of the sum in (3.4) and in the fact that this expression is actually a discrete sum: recall that the first three-point function in this equation involves l screening operators while the second one involves the remaining $s-l$ ones. In order to obtain an expression for generic unflowed external states we will identify the terms in the sum over l with the residues of a meromorphic function extending the summands sequence. This will allow us to rewrite the four-point correlator as a complex integral where the integer nature of the number of screening operators will be strictly restricted to the choice of the integration contour. For a suitable set of the kinematical parameters this contour can be fixed and generic amplitudes can be obtained. This strategy to perform the analytic continuation of (3.4) to generic spin values of the external states in the semiclassical limit is inspired by [14]. In the next subsection we discuss the validity of the result for arbitrary values of the affine level k.

In order to trade the sum in (3.4) for an integral, let us first notice that it can be freely extended to ∞ (see (3.2)). Furthermore, given that the two-point function in the denominator of (3.4) diverges as $\Gamma(0)$, we can use the following identity, which is valid for any sequence $K(l)$:

$$\frac{1}{\Gamma(0)}\sum_{l=0}^{\infty} K(l) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathcal{K}(x)\, dx, \tag{3.6}$$

where $\mathcal{K}(x)$ is a meromorphic continuation of $K(l)$ having simple poles at $x = 0, 1, 2, \ldots, \infty$, with $K(x)$ behaving as $\Gamma(-x)$ near them.[12] The contour $\mathcal{C}$ is understood to enclose only these poles and neither of the other poles that $\mathcal{K}(x)$ could have.

Our first task is to find a proper analytic continuation for the sequence of summands in (3.4). To this aim, recall that it was proved in [1] that the Coulomb gas representation of the three-point functions $A_3^{w=0}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix}$ admits such analytic continuation in the number of screening operators leading to the following exact expression [2, 17]:

$$\mathcal{A}_3^{w=0}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \delta^2\left(m_1+m_2+m_3\right) D(-1-j_1, -1-j_2, -1-j_3) W\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix},$$

where we have introduced the $\delta^2\left(m_1+m_2+m_3\right)$ in order to reinforce the conservation law implicit in the free field computation, $D(j_1, j_2, j_3)$ is the structure constant given by

$$D(j_1, j_2, j_3) = \frac{G(1+j_1+j_2+j_3)G(j_1+j_2-j_3)G(j_2+j_3-j_1)G(j_3+j_1-j_2)}{\nu^{-j_1-j_2-j_3-1}G_0 G(1+2j_1)G(1+2j_2)G(1+2j_3)}, \tag{3.8}$$

with

$$G(j) = (k-2)^{\frac{j(1-j-k)}{2(k-2)}}\,\Gamma_2(-j|1,k-2)\,\Gamma_2(k-1+j|1,k-2),$$

$\Gamma_2(x|1,w)$ being the Barnes double Gamma function, $G_0 = -2\pi^2\gamma\left(1-\rho\right)G(-1)$ and

$$W\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \int d^2x_1\, d^2x_2\, x_1^{j_1+m_1}\overline{x}_1^{j_1+\overline{m}_1} x_2^{j_2+m_2}\overline{x}_2^{j_2+\overline{m}_2} |1-x_1|^{-2j_{13}-2}|1-x_2|^{-2j_{23}-2}|x_1-x_2|^{-2j_{12}-2}.$$

[12] Eq. (3.6) is a suitable form of the classical Nörlund-Rice theorem for infinite sums, which states that

$$\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}H(l) = \frac{1}{2\pi i}\oint_{\mathcal{C}}\Gamma(-x)\mathcal{H}(x)\,dx\,, \tag{3.7}$$

for any meromorphic continuation $\mathcal{H}(x)$ of $H(l)$ having no poles in the positive integer numbers. Eq. (3.6) is obtained from (3.7) after using the formal expression $l! = \Gamma(1+l) = (-1)^l\Gamma(0)/\Gamma(-l)$.

This function was computed in [35] and it was shown in [17] that it reduces to

$$W_1\begin{bmatrix} j_1,j_2,j_3\\ m_1,m_2,m_3\end{bmatrix} = (-1)^{\overline{m}_3-m_3+\overline{q}_1}\pi^2\frac{\gamma(-1-j_1-j_2-j_3)\gamma(1+2j_1)}{\gamma(1+j_{12})\gamma(1+j_{13})}\frac{\Gamma(1+j_2-m_2)}{\Gamma(-j_2+\overline{m}_2)}\frac{\Gamma(1+j_3-m_3)}{\Gamma(-j_3+\overline{m}_3)}$$
$$\times\left\{\frac{\Gamma(1+j_3+m_3)}{\Gamma(1+j_3+m_3-q_1)}{}_3F_2\left[\begin{matrix}-q_1,\,-j_{12},\,1+j_{23}\\ -2j_1,\,1+j_3+m_3-q_1\end{matrix}\middle|1\right]\times\text{c.c.}\right\}, \tag{3.9}$$

for $m_1 = j_1 - q_1$ and $\overline{m}_1 = j_1 - \overline{q}_1$ with $q_1, \overline{q}_1 = 0, 1, 2, \ldots$.

In order to analyze the analytic structure of the summand in (3.4), it is useful to parametrize both three-point functions in a similar way by using the following identity:

$$\mathcal{A}_3^{w=0}\begin{bmatrix} j_3,j_4,-1-j\\ m_3,m_4,m\end{bmatrix}\mathcal{A}_2^{w=0}\begin{bmatrix} j,-1-j\\ -m,m\end{bmatrix}^{-1} = \mathcal{A}_3^{w=0}\begin{bmatrix} j_3,j_4,j\\ m_3,m_4,m\end{bmatrix}\mathcal{A}_2^{w=0}\begin{bmatrix} j,j\\ -m,m\end{bmatrix}^{-1},$$

which follows from (2.6)-(2.7) and (3.5) in [17], so that, up to an irrelevant factor, we can rewrite (3.4) as the following integral:

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1,j_2,j_3,j_4\\ m_1,m_2,m_3,m_4\end{bmatrix} \tag{3.10}$$
$$= \oint_{\mathcal{C}} |z|^{2(\Delta_j-\Delta_{j_1}-\Delta_{j_2})}\mathcal{A}_3^{w=0}\begin{bmatrix} j_1,j_2,j\\ m_1,m_2,-m\end{bmatrix}\mathcal{A}_3^{w=0}\begin{bmatrix} j_3,j_4,j\\ m_3,m_4,m\end{bmatrix}\mathcal{A}_2^{w=0}\begin{bmatrix} j,j\\ -m,m\end{bmatrix}^{-1} dj,$$

where $\mathcal{C}$ encloses the poles at $j = -1 - j_1 - j_2 + \mathbb{Z}_{\geq 0}$.

Some care must be taken when applying (3.6) since it is valid for a meromorphic extension behaving as a gamma function near the integer poles. The three-point function $\mathcal{A}_3^{w=0}\begin{bmatrix} j_1,j_2,j\\ m_1,m_2,-m\end{bmatrix}$ exhibits this behaviour near the poles at $j = -1 - j_1 - j_2 + \mathbb{Z}_{\geq 0}$ in the factor $\gamma(-1 - j_1 - j_2 - j)$ in (3.9). The structure constants have no such poles.

The fact that s is an integer number plays no role in (3.10) and, in that sense, this expression can be thought to be valid even for $j_1 + j_2 + j_3 + j_4 + 1 \notin \mathbb{N}_0$. Recall that the three-point functions involve fields with generic kinematical configurations. However, it is important to notice that although the spins of the external states are no longer restricted, the integer nature of the number of screening operators remains encoded in the prescription for the choice of the integration contour. Indeed, it is necessary to specify $\mathcal{C}$ in order to have a well-defined analytic continuation. This does not seem possible for arbitrary configurations [14], but in the semiclassical limit, one can freely set $\mathcal{C} = \mathcal{P} = -1/2 + i\mathbb{R}$ restricting the quantum numbers of the external states as follows:

$$\begin{cases} |\mathrm{Re}(j_1+j_2+1)| < \frac{1}{2}, & |\mathrm{Re}(j_3+j_4+1)| < \frac{1}{2},\\ |\mathrm{Re}(j_1-j_2)| < \frac{1}{2}, & |\mathrm{Re}(j_3-j_4)| < \frac{1}{2}, \end{cases} \tag{3.11}$$

$$\begin{cases} \max\{m_1+m_2, \overline{m}_1+\overline{m}_2\} > -\frac{1}{2},\\ \min\{m_1+m_2, \overline{m}_1+\overline{m}_2\} < \frac{1}{2}. \end{cases} \tag{3.12}$$

Indeed, taking the $k \to \infty$ limit, the poles of the first three-point function in (3.10) are located at

$$\begin{cases} j = -1 - j_1 - j_2 + \mathbb{Z}_{\geq 0}, \\ j = j_2 - j_1 + \mathbb{Z}_{\geq 0}, \end{cases} \tag{3.13}$$

$$\begin{cases} j = j_1 + j_2 - \mathbb{Z}_{\geq 0}, \\ j = j_1 - j_2 - 1 - \mathbb{Z}_{\geq 0}, \end{cases} \tag{3.14}$$

$$j = -\max\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} - \mathbb{Z}_{>0}\,,$$

while those coming from the second three-point function are placed at

$$\begin{cases} j = -1 - j_3 - j_4 + \mathbb{Z}_{\geq 0}, \\ j = j_4 - j_3 + \mathbb{Z}_{\geq 0}, \end{cases} \tag{3.15}$$

$$\begin{cases} j = j_3 + j_4 - \mathbb{Z}_{\geq 0}, \\ j = j_3 - j_4 - 1 - \mathbb{Z}_{\geq 0}, \end{cases} \tag{3.16}$$

$$j = \min\{m_1 + m_2, \overline{m}_1 + \overline{m}_2\} - \mathbb{Z}_{>0}\,.$$

Under (3.12) the poles depending on m_1, m_2 and $\overline{m}_1, \overline{m}_2$ lie in the left half complex $j-$plane, and this is also the case for the poles at (3.14) and (3.16). It follows that closing the contour at infinity to the right, the only poles encircled are at (3.13) and (3.15). By virtue of the parametrization (3.5) it is easy to see that the contributions from the residues in both families of poles are identical. Finally, let us notice that both series of poles in (3.13) are related by the reflection $j_2 \leftrightarrow (-1 - j_2)$. It is proved in appendix A.2 that the residues at the second series of poles in (3.13) vanish if the state at $z_2, \overline{z}_2 = 1$ lies in a discrete series.

Summarizing, we have found, up to irrelevant factors, that the leading term in the factorization of the four-point function is given, in the semiclassical limit, by

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{bmatrix} = \tag{3.17}$$
$$\int_{\mathcal{P}} dj\ |z|^{2(\Delta_j - \Delta_{j_1} - \Delta_{j_2})} \mathcal{A}_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ m_1, m_2, -m \end{bmatrix} \mathcal{A}_3^{w=0}\begin{bmatrix} j_3, j_4, j \\ m_3, m_4, m \end{bmatrix} \mathcal{A}_2^{w=0}\begin{bmatrix} j, j \\ -m, m \end{bmatrix}^{-1}.$$

This expression makes no reference at all to the integer nature of the number of screening operators and it is valid for external states restricted as in (3.11), (3.12). For other values of the kinematical parameters it must be defined by analytic continuation, as discussed in [2, 3].

Eq. (3.18) agrees with the one obtained by transforming the x-basis integral formula (1.2) for the four-point function to the m-basis.[13] In the H_3^+-WZNW model, (1.2) was obtained in [14] in the mini-superspace limit, which describes a semiclassical region of the full theory, and it was postulated to be valid for generic values of k from the OPE of normalizable states and the factorization ansatz in [2, 3]. Here, we have deduced it

[13] See [30] for this computation and [36] for an alternative representation of the integral transform of (1.2) to the m-basis.

also in the large-k limit using the Coulomb gas method. If w-conserving amplitudes in the AdS_3-WZNW models are related by analytic continuation to correlators of the H_3^+-WZNW model, as it is widely believed, a similar postulate would allow to extend the validity of (3.18) beyond the semiclassical regime. Such conjecture however is more subtle in the Lorentzian model than in its Euclidean counterpart due to the spectral flow representations. In the following subsection we discuss this issue.

3.3 Factorization into spectral flow violating three-point functions

The OPE of normalizable states determining the four-point functions (1.2) in the H_3^+-WZNW model would give an incorrect zero answer if used to compute w-violating three-point functions in the AdS_3-WZNW model. Actually, consistency with the spectral flow selection rules leads to the following OPE for spectral flow images of primary fields in the AdS_3-WZNW model [30]:

$$\Phi^{j_1,w_1}_{m_1,\overline{m}_1}(z_1,\overline{z}_1)\Phi^{j_2,w_2}_{m_2,\overline{m}_2}(z_2,\overline{z}_2) = \\ \sum_{w=0,\pm1}\int_{\mathcal{P}} \mathcal{A}_3^w \begin{bmatrix} j_1\,, j_2, \,-1-j_3 \\ m_1, m_2, -m_3 \end{bmatrix} z_{12}^{-\hat{\Delta}_{12}} \overline{z}_{12}^{-\hat{\overline{\Delta}}_{12}} \Phi^{j_3,w_3}_{m_3,\overline{m}_3}(z_2,\overline{z}_2)\, dj_3 + \cdots\,, \tag{3.18}$$

where $w = w_3 - w_1 - w_2$, $z_{12} = z_1 - z_2$, $\hat{\Delta}_{12} = \hat{\Delta}_{j_1,m_1,w_1} + \hat{\Delta}_{j_2,m_2,w_2} - \hat{\Delta}_{j_3,m_3,w_3}$ and

$$\mathcal{A}_3^{w=\pm1}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \\ \frac{\delta^2\,(m_1+m_2+m_3 \mp k/2)}{\gamma(j_1+j_2+j_3+3-k/2)} \widetilde{D}(-1-j_1,-1-j_2,-1-j_3)\widetilde{W}\begin{bmatrix} j_1, j_2, j_3 \\ \mp m_1, \mp m_2, \mp m_3 \end{bmatrix}$$

with

$$\widetilde{D}(j_1,j_2,j_3) \sim B(j_1) D\left(-\frac{k}{2} - j_1, j_2, j_3\right),$$

up to k-dependent, j-independent factors and

$$\widetilde{W}\begin{bmatrix} j_1\,,\, j_2\,,\, j_3 \\ m_1, m_2, m_3 \end{bmatrix} = \frac{\Gamma(1+j_1+m_1)}{\Gamma(-j_1-\overline{m}_1)} \frac{\Gamma(1+j_2+\overline{m}_2)}{\Gamma(-j_2-m_2)} \frac{\Gamma(1+j_3+\overline{m}_3)}{\Gamma(-j_3-m_3)}.$$

Although it is necessary to further truncate this OPE in order to avoid inconsistencies with the spectral flow symmetry, the physical mechanism determining the decoupling not being yet completely understood, several successful checks have been performed on the fusion rules obtained from (3.18). In particular, they reproduce the classical tensor product of representations of $\mathrm{SL}(2,\mathbb{R})$ in the $k \to \infty$ limit and moreover, for generic $k > 2$ they establish the closure of the operator algebra on the Hilbert space of the AdS_3-WZNW model determined in [16].

The factorization ansatz based on this OPE would give an expression for the w-conserving four-point correlation functions involving both spectral flowed and unflowed intermediate states. This conclusion also follows from the results in [4], where $w = 1$ long strings were found in the s-channel factorization of the four-point functions of $w = 0$ short strings starting from (1.2), rewriting the integrand and moving the integration contour. However, these observations pose an apparent contradiction with (3.2).

To understand this issue, recall that we have performed the Coulomb gas computation of the expectation value of four unflowed vertices without any insertion of spectral flow operators, namely, we have taken $N_+ = N_- = 0$ in (2.7). Therefore we should not expect to be able to recognize w-violating three-point functions in the factorization limit since $w \neq 0$ amplitudes require insertions of vertices $\eta^{\pm}$. However, the full final result for the w-conserving four-point function must be the same, independently of the (even) number of these insertions because they simply act as picture changing operators. This suggests either that there are no w-violating channels or that both channels give equivalent expansions. These two possibilities also follow if correlation functions in the AdS_3-WZNW model are to be obtained by analytic continuation from those in the H_3^+-WZNW model, because the spectral flow fields do not belong to the spectrum of the Euclidean theory. The results in [4, 30] force the conclusion that both channels give equivalent contributions. However, we should not expect to be able to verify this equivalence in general just by looking at the leading terms in the factorization limit. Rather, since the spectral flow operation maps primaries into non-primaries, a general proof of this statement would require making explicit the higher order terms in (3.18) and possibly some contour manipulations.

Despite these general arguments, in the remaining of this section we use the Coulomb gas approach to illustrate in a particular example the assertion that products of w-preserving and violating three-point functions give the same contributions to the w-conserving four-point functions.

Let us start by evaluating the following amplitude

$$A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \\ w_1, w_2, w_3, w_4 \end{bmatrix} = \Gamma(-s)\left\langle V^{j_1,w_1}_{m_1=\overline{m}_1=j_1}(0) V^{j_2,w_2}_{m_2,\overline{m}_2}(z) V^{j_3,w_3}_{m_3=\overline{m}_3=-j_3}(1) V^{j_4,w_4}_{m_4,\overline{m}_4}(+\infty)\eta^-(\zeta^-)\eta^+(\zeta^+)\mathcal{Q}^s\right\rangle . \tag{3.19}$$

The insertion of the spectral flow operators will be explained later.

After performing the corresponding field contractions we get

$$\begin{aligned} A_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \\ w_1, w_2, w_3, w_4 \end{bmatrix} = \frac{\Gamma(-s)}{\pi^2\Gamma(0)^2}&\Big[(\zeta^- - z)^{-(j_2-m_2)}(1-\zeta^-)^{-2j_3}(\zeta^+ - z)^{-(j_2+m_2)} \\ &\times(\zeta^- - \zeta^+)^{-(k-1)}(\zeta^+)^{-2j_1} z^{2j_1j_2\rho-\frac{k}{2}w_1w_2-w_1m_2-w_2j_1} \\ &\times(1-z)^{2j_2j_3\rho-\frac{k}{2}w_2w_3+w_2j_3-w_3m_2}\times \text{c.c}\Big] \\ &\times\int\prod_{i=1}^{s} d^2y_i|y_i|^{-4j_1\rho}|z-y_i|^{-4j_2\rho}|1-y_i|^{-4j_3\rho}|\zeta^+ - y_i|^4 \\ &\times\prod_{i<j}^{s}|y_i-y_j|^{4\rho}\left[\frac{1}{\mathcal{P}}\partial_1\ldots\partial_s(\Lambda\mathcal{P})\times \text{c.c}\right], \end{aligned}$$

where we have defined

$$\mathcal{P}=\prod_{i=1}^{s}(z-y_i)^{-(j_2-m_2)}(1-y_i)^{-2j_3}(\zeta^+-y_i)^{-(k-2)}\prod_{i<j}(y_i-y_j)\quad\text{and}\quad\Lambda=\prod_{i=1}^{s}\frac{\zeta^--y_i}{\zeta^+-y_i}.$$

It was shown in [1] that this expression reproduces the one obtained using the prescription introduced in [32] to compute correlators involving spectral flowed operators. Therefore, the dependence on $\zeta^\pm$ and $\overline{\zeta}^\pm$ cancels and we can freely take[14] $\zeta^-=\overline{\zeta}^-=0$ and $\zeta^+=\overline{\zeta}^+=1$, obtaining

$$\begin{aligned}
&A_4^{w=0}\begin{bmatrix} j_1,j_2,j_3,j_4\\ j_1,m_2,-j_3,m_4\\ w_1,w_2,w_3,w_4\end{bmatrix}\\
&=\frac{\Gamma(-s)}{\pi^2\Gamma(0)^2}\left[z^{2j_1j_2\rho-\frac{k}{2}w_1w_2-w_1m_2-w_2j_1-j_2+m_2}(1-z)^{2j_2j_3\rho-\frac{k}{2}w_2w_3+w_2j_3-w_3m_2-j_2-m_2}\times\text{c.c}\right]\\
&\quad\times\int\prod_{i=1}^{s}d^2y_i|y_i|^{-4j_1\rho+2}|z-y_i|^{-4j_2\rho}|1-y_i|^{-4j_3\rho+2}\prod_{i<j}|y_i-y_j|^{4\rho}\left[\frac{1}{\mathcal{P}'}\partial_1\ldots\partial_s\mathcal{P}'\times\text{c.c}\right]
\end{aligned}$$

where

$$\mathcal{P}'=\prod_{i=1}^{s}y_i(z-y_i)^{-(j_2-m_2)}(1-y_i)^{-2j_3-k+1}\prod_{i<j}(y_i-y_j).$$

It is easy to check that this expression equals, up to the factor $(\pi\Gamma(0))^{-2}$, the Coulomb integral realization of the amplitude $A_4^{w=0}\begin{bmatrix} -1-\tilde{j}_1,j_2,-1-\tilde{j}_3,j_4\\ -\tilde{j}_1,\ m_2,\ \ \tilde{j}_3,\ \ m_4\\ w_1-1,w_2,w_3+1,w_4\end{bmatrix}$, where we have introduced

$$\tilde{j}_1=-\frac{k}{2}-j_1,\qquad\qquad \tilde{j}_3=-\frac{k}{2}-j_3\,.$$

Notice that the conservation laws for this correlation function, when no spectral flow operators are inserted, reproduce those of (3.20). Indeed, the spectral flow operators were inserted in (3.20) to achieve this equality.

Using the reflection identity [3, 33, 37] and $c^{\tilde{j}}_{\pm\tilde{j},\pm\tilde{j}}=\pi\Gamma(0)$, we can write

$$V^{-1-\tilde{j},w}_{\pm\tilde{j},\pm\tilde{j}}=B(\tilde{j})\,c^{\tilde{j}}_{\pm\tilde{j},\pm\tilde{j}}V^{\tilde{j},w}_{\pm\tilde{j},\pm\tilde{j}}=\pi\Gamma(0)B(\tilde{j})V^{\tilde{j},w}_{\pm\tilde{j},\pm\tilde{j}},$$

and then it is straightforward to show that

$$\begin{aligned}
A_4^{w=0}\begin{bmatrix} j_1,j_2,j_3,j_4\\ j_1,m_2,-j_3,m_4\\ w_1,w_2,w_3,w_4\end{bmatrix}&=B(\tilde{j}_1)B(\tilde{j}_3)A_4^{w=0}\begin{bmatrix} \tilde{j}_1, & j_2, & \tilde{j}_3, & j_4\\ -\tilde{j}_1, & m_2, & \tilde{j}_3, & m_4\\ w_1-1, & w_2, & w_3+1, & w_4\end{bmatrix} \qquad (3.20)\\
&=z^{m_2+\frac{k}{2}w_2}\overline{z}^{\overline{m}_2+\frac{k}{2}w_2}(1-z)^{-m_2-\frac{k}{2}w_2}(1-\overline{z})^{-\overline{m}_2-\frac{k}{2}w_2}B(\tilde{j}_1)B(\tilde{j}_3)A_4^{w=0}\begin{bmatrix} \tilde{j}_1,j_2,\tilde{j}_3,j_4\\ -\tilde{j}_1,m_2,\tilde{j}_3,m_4\\ w_1,w_2,w_3,w_4\end{bmatrix}.
\end{aligned}$$

[14] Notice that this can be done only because there is a highest-weight state inserted at $z_1,\overline{z}_1=0$ and a lowest-weight state at $z_3,\overline{z}_3=1$.

This identity was assumed in [30] as the starting point of the proof that products of w-conserving or w-violating three-point functions give the same contribution when factorizing these four-point functions. To be more explicit, let us show this statement in the particular case $w_i = 0$, $i = 1, \dots, 4$.

On the one hand, we have already found that (see eq. (3.4) and the discussion below)

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \end{bmatrix} = \sum_{l=0}^{s} |z|^{2(\Delta_j - \Delta_{j_1} - \Delta_{j_2})} A_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ j_1, m_2, -m \end{bmatrix} A_3^{w=0}\begin{bmatrix} j_3, j_4, j \\ -j_3, m_4, m \end{bmatrix} A_2^{w=0}\begin{bmatrix} j, j \\ -m, m \end{bmatrix}^{-1} \tag{3.21}$$

where $j = -1 - j_1 - j_2 + l$, and we showed that this expression can be analytically continued as

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \end{bmatrix} = \int_{\mathcal{P}} dj\ |z|^{2(\Delta_j - \Delta_{j_1} - \Delta_{j_2})} \mathcal{A}_3^{w=0}\begin{bmatrix} j_1, j_2, j \\ j_1, m_2, -m \end{bmatrix} \mathcal{A}_3^{w=0}\begin{bmatrix} j_3, j_4, j \\ -j_3, j_4, m \end{bmatrix} \mathcal{A}_2^{w=0}\begin{bmatrix} j, j \\ -m, m \end{bmatrix}^{-1} \tag{3.22}$$

for configurations of the external states lying in (3.11)–(3.12).

On the other hand, from (3.21) we obtain

$$\begin{aligned}\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \end{bmatrix} &= B(\tilde{j}_1) B(\tilde{j}_3) z^{m_2} \overline{z}^{\overline{m}_2} \sum_{l=0}^{s} |z|^{2(\Delta_{\tilde{j}} - \Delta_{\tilde{j}_1} - \Delta_{j_2})} \\ &\quad \times A_3^{w=0}\begin{bmatrix} \tilde{j}_1, j_2, \tilde{j} \\ -\tilde{j}_1, m_2, -\tilde{m} \end{bmatrix} A_3^{w=0}\begin{bmatrix} \tilde{j}_3, j_4, \tilde{j} \\ \tilde{j}_3, m_4, \tilde{m} \end{bmatrix} A_2^{w=0}\begin{bmatrix} \tilde{j}, \tilde{j} \\ -\tilde{m}, \tilde{m} \end{bmatrix}^{-1},\end{aligned} \tag{3.23}$$

where $\tilde{j} = -1 - \tilde{j}_1 - j_2 + l$.

Following the procedure leading to (3.21) in the case of the three-point functions, one can show from the Coulomb integral expressions that the factors $B(\tilde{j}_1)$ and $B(\tilde{j}_3)$ can be reabsorbed as

$$B(\tilde{j}_1) A_3^{w=0}\begin{bmatrix} \tilde{j}_1, j_2, \tilde{j} \\ -\tilde{j}_1, m_2, -\tilde{m} \end{bmatrix} = A_3^{w=1}\begin{bmatrix} j_1, j_2, \tilde{j} \\ j_1, m_2, -\tilde{m} \end{bmatrix},$$

and a similar expression for the second three-point function in (3.24). Since the coordinate independent coefficient of the two-point functions of states in different spectral flow sectors does not change, we finally get the following expression:

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \end{bmatrix} = \sum_{l=0}^{s} |z|^{2(\hat{\Delta}_{\tilde{j},\tilde{m},w=-1} - \Delta_{j_1} - \Delta_{j_2})} A_3^{w=1}\begin{bmatrix} j_1, j_2, \tilde{j} \\ j_1, m_2, -\tilde{m} \end{bmatrix} A_3^{w=-1}\begin{bmatrix} j_3, j_4, \tilde{j} \\ -j_3, m_4, \tilde{m} \end{bmatrix} A_2^{w=0}\begin{bmatrix} \tilde{j}, \tilde{j} \\ -\tilde{m}, \tilde{m} \end{bmatrix}^{-1}, \tag{3.24}$$

the factor $z^{m_2}\,\overline{z}^{\overline{m}_2}$ in (3.24) being needed in order to reproduce the correct conformal weight of the intermediate states.

The equivalence between expansions of the same four-point function in terms of either w-conserving or w-violating three-point functions can be seen comparing eqs. (3.22) and (3.25).

We mentioned above that the spectral flow makes the validity of (3.18) beyond the semiclassical limit in the AdS_3 model more subtle than in the H_3^+ model. But the possibility of encoding the unflowed contributions in terms of spectral flowed intermediate states supports the conjecture that (3.18) also holds for finite values of the affine level.[15] If this is the case, starting from (3.23) instead of (3.22) would lead us to the following analytic continuation of (3.25):[16]

$$\mathbb{A}_4^{w=0}\begin{bmatrix} j_1, j_2, j_3, j_4 \\ j_1, m_2, -j_3, m_4 \end{bmatrix}$$
$$= \int_{\mathcal{P}} dj\ |z|^{2(\hat{\Delta}_{\tilde{j},\tilde{m},\tilde{w}=-1}-\Delta_{j_1}-\Delta_{j_2})} A_3^{w=1}\begin{bmatrix} j_1, j_2, \tilde{j} \\ j_1, m_2, -\tilde{m} \end{bmatrix} \mathcal{A}_3^{w=-1}\begin{bmatrix} j_3, j_4, \tilde{j} \\ -j_3, m_4, \tilde{m} \end{bmatrix} \mathcal{A}_2^{w=0}\begin{bmatrix} \tilde{j}, \tilde{j} \\ -\tilde{m}, \tilde{m} \end{bmatrix}^{-1}.$$

Indeed, the equivalence of this last expression and (3.23) was obtained in [30] and it was the starting point for an explicit verification that the truncation imposed on the operator algebra by the spectral flow symmetry is realized at the level of physical amplitudes.

4 Summary and discussion

We have computed $w-$conserving four-point correlation functions on the sphere in the AdS_3-WZNW model using the Coulomb gas approach. The requirement of integer numbers of screening operators, a well known shortcoming of the formalism for models with continuous sets of fields, demands considering operators with quantized values of the sum of their spins, implying that only expectation values of certain states in discrete representations can be evaluated without performing any analytic continuation. The result in this case was obtained as the monodromy invariant sum of products of holomorphic and antiholomorphic conformal blocks, namely, equation (2.37).

The full integral expression for the conformal blocks presented in (2.26) extends previous results obtained in [19] where a simplified setting, sufficient to derive the operator algebra, was considered, namely, two highest- and two lowest-weight operators with $j_1 = j_4$ and $j_2 = j_3$. Indeed, we have computed the $\beta-\gamma$ ghost contributions for generic configurations of fields, only restricted by the assumption of one highest-weight state and the requirement of arbitrary positive integer numbers of screenings. To this aim, the explicit computation of the ghost system involved in the three-point functions that was presented

[15] Additional indications that (3.18) holds beyond the semiclassical limit are given by the fact that the OPE leading to this expression in the bootstrap approach to the H_3^+-WZNW model implemented in [2] reproduces the well-known fusion rules of admissible degenerate representations and by the results on the structure of the factorization of string theory on AdS_3 in [4].

[16] The analytic continuation leading to (3.23) cannot be directly implemented in the semiclassical limit for (3.25).

in [1], although not strictly necessary to obtain the Clebsch-Gordan coefficients, turned out to be a useful starting point to address the computation of these higher-point functions. Actually, unlike the case of the three-point functions, where the full form of the kinematical factor follows from the SL(2) space-time symmetry of the model, the ghost correlators give a non-trivial dependence on the coordinates to the conformal blocks of the four-point amplitudes.

As discussed in section 2, relaxing the highest-weight condition assumed for one of the operators turns the elementary symmetric polynomials in (2.20) into Schur polynomials and the coefficients given in (2.17) must be replaced by the most general expression (A.5). Although the identities (2.24), (2.30) and (2.33) that we have derived for the elementary symmetric polynomials cannot be easily extended to the general case, the $z, \overline{z} \to 0$ limit of the integrals (2.12) has been computed in [34] and then the conformal blocks for four generic states can be reconstructed along similar steps as those we have followed in subsection 2.3. In any case, we have shown in section 3 that the leading terms in the factorization limit can be identified with products of three-point functions and then, from (3.4) it is straightforward to see that the Selberg integrals in (2.37) must be replaced by combinations of Aomoto integrals in the amplitude involving four global descendants or their spectral flow images with spin values adding up to an integer number.

Besides the ambiguities involved in the analytic continuation needed to obtain amplitudes of arbitrary external states, the Coulomb gas method suffers from the disadvantage of requiring quite a bit of tedious algebra if compared to the bootstrap approach. Nevertheless, despite these problems, we were able to present an alternative derivation of the expression obtained for the four-point functions in [2, 3, 14]. Indeed, we have shown that for special configurations of fields, the semiclassical limit of the leading terms of the four-point function may be rewritten as an integral over the spins of the intermediate states. Actually, the expression (3.18) obtained in section 3 is valid for fields satisfying (3.11) and (3.12) and for other values of the kinematical parameters it must be defined by the analytic continuation discussed in [3], *i.e.*, the large-k limit of the leading terms of the amplitude are given by (3.18) plus the contributions of all the poles that cross the integration contour. This result reproduces in the $m-$basis the $x-$basis amplitude for the H_3^+ model obtained in [14] in the mini-superspace approximation. The explicit Coulomb gas calculation that we have presented here can thus be considered as an independent check of the factorization ansatz based on the OPE of normalizable primary fields proposed in [2, 3] for the H_3^+-WZNW model.

The procedure followed in section 3 to convert the discrete sums into the integral expression (3.18) provides a possible resolution of the problem raised in [19] regarding the ambiguity involved in the analytic continuation of amplitudes containing states with rational spin values in the SU(2) CFT. Moreover, it gives an alternative route to the use of the fractional calculus introduced in [21]. Furthermore, since (3.18) was directly deduced in the $m-$basis, it gives support to the process implemented in [30, 36] to transform (1.2) from the $x-$basis and helps clarify the related questions raised in the introduction about exchanging the order of summation and integration as well as convergence of the integral transform.

Following [4, 30], we have argued that the factorization into products of spectral

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flow preserving or violating three-point functions gives equivalent contributions to the w-conserving four-point functions and we proved this statement in a particular set of amplitudes using the Coulomb gas realization. Indeed, we have shown that the leading terms in the factorization limit of the discrete sums in (2.37) for certain amplitudes can be rewritten alternatively as a sum of products of two $w = 0$ or of one $w = 1$ and one $w = -1$ three-point functions. This result provides an independent confirmation of the factorization *ansatz* proposed in [30] and of the observation that w-conserving correlators in the AdS_3-WZNW model are to be obtained from correlators in the H_3^+-WZNW model through analytic continuation, although the spectral flow representations are not contained in the spectrum of the Euclidean model. Furthermore, it provides additional support for the proposal that (3.18) also holds for finite k, beyond the semiclassical limit.

Not having achieved a closed expression for the four-point functions cannot be attributed to the method. Neither in the less complicated cases of Liouville theory or H_3^+-WZNW model an explicit closed formula is known. Although important progress has recently been achieved in the former theory through the identification of the conformal blocks with Nekrasov's partition function of certain $\mathcal{N} = 2$ superconformal field theories [5, 6], the available amplitude is decomposed into structure constants and $s-$channel conformal blocks that have to be numerically computed with the techniques developed in [10]. The existence of an interesting explicit formula for generic four-point functions in the AdS_3-WZNW model also seems unlikely.

It would be interesting to extend the procedures developed in this paper to gain more insights into four-point functions and conformal blocks and to start understanding $w-$violating amplitudes in order to solve the AdS_3-WZNW model. Indeed, there are several open problems yet. The equivalence between factorizations involving w-conserving or w-violating three-point functions implies that the OPE (3.18) is actually equivalent to the OPE of normalizable states of the H_3^+-WZNW model proposed in [2, 3] when inserted into w-conserving amplitudes. But the latter OPE leads to vanishing w-violating amplitudes, in contradiction with the spectral flow selection rules and the explicit computations performed in [1, 4]. Moreover, the fusion rules obtained from the H_3^+-OPE by analytic continuation are not closed in the spectrum of the AdS_3-WZNW model and they are not compatible with the identification $\hat{D}_j^{\pm,w} = \hat{D}_{-k/2-j}^{\mp,w\pm1}$ implied by the spectral flow symmetry [17, 30]. Furthermore, the OPE (3.18) has to be truncated in order to avoid inconsistencies with the spectral flow symmetry and the physical mechanism determining the decoupling is still not understood. The computation of w-violating four-point functions might shed some light on these issues. Although it requires the insertion of a spectral flow operator and consequently involves the evaluation of a five-point function, we hope to be able to address this problem in the near future applying the techniques developed in this paper.

We expect that these techniques might also be useful to further deepen our knowledge on properties of non-rational CFTs and methods to deal with them.

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A Appendices

A.1 Evaluation of m-dependent coefficients: the general case

For completeness, in this appendix we compute the coefficients $C_{nr}(z)$ in (2.13) for generic configurations of fields.

Consider the determinant of the matrix $(a_{ij})_{i,j=1}^{p}$ made up by the first p rows and p columns of the matrix $(a_{ij}(z))_{i,j=1}^{s-n}$ with entries defined in (2.14). Let us denote it $d_p(z)$. As $d_p^0(z)$, this is a polynomial in z of degree p but satisfying the following recurrence relation:

$$d_p(z) = \ell_1^{s-n-r+p}(z) d_{p-1}(z) - \ell_2^{s-n-r+p-1}(z)\ell_0^{s-n-r+p}(z) d_{p-2}(z), \tag{A.1}$$

with boundary conditions $d_1(z) = \ell_1^{s-n-r+1}(z)$ and $d_2(z) = \ell_1^{s-n-r+1}(z)\ell_1^{s-n-r+2}(z) - \ell_2^{s-n-r+1}(z)\ell_0^{s-n-r+2}(z)$. To deduce (A.1) we have used that $(a_{ij}(z))_{i,j=1}^{p}$ is a tridiagonal matrix.

It is convenient to introduce the following "shifted" parameters:

$$\begin{cases} \alpha' = -s+n+r+\alpha \\ \beta' = -s+n+r+\alpha+\beta \\ \gamma' = -s+n+r+\alpha+\gamma, \end{cases}$$

and rewrite (A.1) more explicitly as

$$\begin{aligned} d_p(z) = {} & \left[(1-p+\beta') + (1-p+\gamma')z\right] d_{p-1}(z) - (p-2-\beta'-\gamma')(p-1)z d_{p-2}(z) \\ & +\alpha'(\alpha'-\beta'-\gamma'-1)z d_{p-2}(z), \end{aligned} \tag{A.2}$$

with $d_1(z) = \beta' + \gamma' z$ and $d_2(z) = \beta'(\beta'-1) + 2\beta'\gamma' z + \gamma'(\gamma'-1)z^2 + \alpha'(\alpha'-\beta'-\gamma'-1)z$.

For the case $\alpha' = 0$ we have found the solution of this recurrence in (2.16), namely,

$$d_p^0(z) = \frac{\Gamma(\beta'+1)}{\Gamma(\beta'-p+1)}\, {}_2F_1\left[\left.\begin{matrix} -p, -\gamma' \\ \beta'-p+1 \end{matrix}\right| z\right].$$

Defining $d_p(z) = d_p^0(z) + \epsilon_p(z)$ and replacing this into (A.2) we obtain the following recurrence for $\epsilon_p(z)$:

$$\begin{aligned} \epsilon_p(z) = {} & \left[(1-p+\beta') + (1-p+\gamma')z\right] \epsilon_{p-1}(z) \\ & -(p-2+\alpha'-\beta'-\gamma')(p-1-\alpha')z\epsilon_{p-2}(z) + \alpha'(\alpha'-\beta'-\gamma'-1)z d_{p-2}^0, \end{aligned}$$

with $\epsilon_1(z) = 0$ and $\epsilon_2(z) = \alpha'(\alpha'-\beta'-\gamma'-1)z$.

Inductively solving this much simpler recurrence it is possible to show that

$$d_p(z) = \sum_{t=0}^{[p/2]} \binom{p-t}{t} \frac{\Gamma(\alpha'+1)\Gamma(\alpha'-\beta'-\gamma'+t-1)}{\Gamma(\alpha'-t+1)\Gamma(\alpha'-\beta'-\gamma'-1)} d^{[t]}_{p-2t}(z), \tag{A.3}$$

where we have defined

$$d^{[t]}_p(z) = \frac{\Gamma(\beta'-t+1)}{\Gamma(\beta'-t-p+1)}\, {}_2F_1\left[\begin{array}{c} -p, -\gamma'+t \\ \beta'-t-p+1 \end{array}\middle| z\right].$$

Therefore, the correlation function involving four generic states is given by

$$A_4^{w=0}\left[\begin{array}{c} j_1, j_2, j_3, j_4 \\ m_1, m_2, m_3, m_4 \end{array}\right] = \Gamma(-s)|z|^{4j_1j_2\rho}|1-z|^{4j_2j_3\rho} \sum_{n,\overline{n}=0}^{s} \sum_{r,\overline{r}=0}^{s-n} [C_{nr}(z)\times c.c.]\, \mathcal{J}_{nr,\overline{nr}}(z,\overline{z}), \tag{A.4}$$

with

$$\begin{aligned} C_{nr} &= (-1)^{s-n-r} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-s+n+r+1)} \frac{\Gamma(s-\alpha-\beta-\gamma)}{\Gamma(s-n-\alpha-\beta-\gamma)} \\ &\quad \times \sum_{t=0}^{\infty} \binom{s-n-t}{t} \frac{\Gamma(\alpha'+1)\Gamma(\alpha'-\beta'-\gamma'+t-1)}{\Gamma(\alpha'-t+1)\Gamma(\alpha'-\beta'-\gamma'-1)} d^{[t]}_{s-n-2t}(z)\, z^{s-n-r} \end{aligned} \tag{A.5}$$

where we have used, again, that the sum in (A.3) can be set to ∞.

Recall that this expression strictly corresponds to a correlator with integer number of screening operators, i.e., $s = j_1+j_2+j_3+j_4+1 \in \mathbb{N}_0$. The four-point function for general kinematic configurations is assumed to be given by an analytic continuation of (A.4) for non-integer values of s, as discussed in section 3.

A.2 Poles of the three-point function and the reflection symmetry

Let us assume that the three-point function $\mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right]$ has a pole located at $j_3 = f(j_1, j_2)$. It is clear that it could also have a pole at $j_3 = f(j_1, -1-j_2)$ and that

$$\mathrm{Res}_{j_3=f(j_1,-1-j_2)} \mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right] = \left\{\mathrm{Res}_{j_3=f(j_1,j_2)} \mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, -1-j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right]\right\}_{j_2\to -1-j_2} \tag{A.6}$$

From [17] we know that

$$\mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, -1-j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right] = B(j_2)\, c^{j_2}_{m_2,\overline{m}_2} \mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right].$$

Inserting this expression into (A.6) we obtain

$$\begin{aligned} &\mathrm{Res}_{j_3=f(j_1,-1-j_2)} \mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right] \\ &\quad = B(-1-j_2)\, c^{-1-j_2}_{m_2,\overline{m}_2} \left\{\mathrm{Res}_{j_3=f(j_1,j_2)} \mathcal{A}_3^{w=0}\left[\begin{array}{c} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{array}\right]\right\}_{j_2\to -1-j_2}. \end{aligned}$$

This equation displays the relation between the residues of the three-point function associated to poles related by reflection.

If the state inserted at $z_2, \overline{z}_2 = 1$ lies in a discrete series, it follows that $\mathcal{A}_3^{w=0}\begin{bmatrix} j_1, j_2, j_3 \\ m_1, m_2, m_3 \end{bmatrix}$ is actually regular at $j_3 = f(j_1, -1-j_2)$ since in this case $c^{-1-j_2}_{m_2,\overline{m}_2}$ vanishes.

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